Problem 7.1.2 Solution

$X_1, X_2 \dots X_n$ are independent uniform random variables with mean value $\mu_X = 7$ and $\sigma_X^2 = 3$

(a) Since X_1 is a uniform random variable, it must have a uniform PDF over an interval $[a, b]$. From Appendix A, we can look up that $\mu_X = (a+b)/2$ and that $\text{Var}[X] = (b-a)^2/12$. Hence, given the mean and variance, we obtain the following equations for a and b.

$$(b-a)^2/12 = 3 \qquad (a+b)/2 = 7 \tag{1}$$

Solving these equations yields $a = 4$ and $b = 10$ from which we can state the distribution of X.

$$f_X(x) = \begin{cases} 1/6 & 4 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) From Theorem 7.1, we know that

$$\text{Var}[M_{16}(X)] = \frac{\text{Var}[X]}{16} = \frac{3}{16} \tag{3}$$

(c)

$$P[X_1 \ge 9] = \int_9^\infty f_{X_1}(x)\, dx = \int_9^{10} (1/6)\, dx = 1/6 \tag{4}$$

(d) The variance of $M_{16}(X)$ is much less than $\text{Var}[X_1]$. Hence, the PDF of $M_{16}(X)$ should be much more concentrated about $E[X]$ than the PDF of X_1. Thus we should expect $P[M_{16}(X) > 9]$ to be much less than $P[X_1 > 9]$.

$$P[M_{16}(X) > 9] = 1 - P[M_{16}(X) \le 9] = 1 - P[(X_1 + \cdots + X_{16}) \le 144] \tag{5}$$

By a Central Limit Theorem approximation,

$$P[M_{16}(X) > 9] \approx 1 - \Phi\left(\frac{144 - 16\mu_X}{\sqrt{16}\sigma_X}\right) = 1 - \Phi(2.66) = 0.0039 \tag{6}$$

As we predicted, $P[M_{16}(X) > 9] \ll P[X_1 > 9]$.

Problem 7.1.3 Solution

This problem is in the wrong section since the *standard error* isn't defined until Section 7.3. However is we peek ahead to this section, the problem isn't very hard. Given the sample mean estimate $M_n(X)$, the standard error is defined as the standard deviation $e_n = \sqrt{\text{Var}[M_n(X)]}$. In our problem, we use samples X_i to generate $Y_i = X_i^2$. For the sample mean $M_n(Y)$, we need to find the standard error

$$e_n = \sqrt{\text{Var}[M_n(Y)]} = \sqrt{\frac{\text{Var}[Y]}{n}}. \tag{1}$$

Since X is a uniform $(0,1)$ random variable,

$$E[Y] = E\left[X^2\right] = \int_0^1 x^2\,dx = 1/3, \tag{2}$$

$$E\left[Y^2\right] = E\left[X^4\right] = \int_0^1 x^4\,dx = 1/5. \tag{3}$$

Thus $\text{Var}[Y] = 1/5 - (1/3)^2 = 4/45$ and the sample mean $M_n(Y)$ has standard error

$$e_n = \sqrt{\frac{4}{45n}}. \tag{4}$$

Problem 7.2.2 Solution

We know from the Chebyshev inequality that

$$P\left[|X - E[X]| \geq c\right] \leq \frac{\sigma_X^2}{c^2} \tag{1}$$

Choosing $c = k\sigma_X$, we obtain

$$P\left[|X - E[X]| \geq k\sigma\right] \leq \frac{1}{k^2} \tag{2}$$

The actual probability the Gaussian random variable Y is more than k standard deviations from its expected value is

$$P\left[|Y - E[Y]| \geq k\sigma_Y\right] = P\left[Y - E[Y] \leq -k\sigma_Y\right] + P\left[Y - E[Y] \geq k\sigma_Y\right] \tag{3}$$

$$= 2P\left[\frac{Y - E[Y]}{\sigma_Y} \geq k\right] \tag{4}$$

$$= 2Q(k) \tag{5}$$

The following table compares the upper bound and the true probability:

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
Chebyshev bound	1	0.250	0.111	0.0625	0.040
$2Q(k)$	0.317	0.046	0.0027	6.33×10^{-5}	5.73×10^{-7}

(6)

The Chebyshev bound gets increasingly weak as k goes up. As an example, for $k = 4$, the bound exceeds the true probability by a factor of 1,000 while for $k = 5$ the bound exceeds the actual probability by a factor of nearly 100,000.

Problem 7.2.4 Solution

On each roll of the dice, a success, namely snake eyes, occurs with probability $p = 1/36$. The number of trials, R, needed for three successes is a Pascal $(k = 3, p)$ random variable with

$$E[R] = 3/p = 108, \qquad \mathrm{Var}[R] = 3(1-p)/p^2 = 3780. \tag{1}$$

(a) By the Markov inequality,

$$P[R \geq 250] \leq \frac{E[R]}{250} = \frac{54}{125} = 0.432. \tag{2}$$

(b) By the Chebyshev inequality,

$$\begin{aligned} P[R \geq 250] = P[R - 108 \geq 142] &= P[|R - 108| \geq 142] \qquad (3)\\ &\leq \frac{\mathrm{Var}[R]}{(142)^2} = 0.1875. \qquad (4) \end{aligned}$$

(c) The exact value is $P[R \geq 250] = 1 - \sum_{r=3}^{249} P_R(r)$. Since there is no way around summing the Pascal PMF to find the CDF, this is what `pascalcdf` does.

```
>> 1-pascalcdf(3,1/36,249)
ans =
    0.0299
```

Thus the Markov and Chebyshev inequalities are valid bounds but not good estimates of $P[R \geq 250]$.

Problem 7.3.1 Solution

For an an arbitrary Gaussian (μ, σ) random variable Y,

$$\begin{aligned} P[\mu - \sigma \leq Y \leq \mu + \sigma] &= P[-\sigma \leq Y - \mu \leq \sigma] \qquad (1)\\ &= P\left[-1 \leq \frac{Y - \mu}{\sigma} \leq 1\right] \qquad (2)\\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827. \qquad (3) \end{aligned}$$

Note that Y can be any Gaussian random variable, including, for example, $M_n(X)$ when X is Gaussian. When X is not Gaussian, the same claim holds to the extent that the central limit theorem promises that $M_n(X)$ is nearly Gaussian for large n.

Problem 7.3.4 Solution

(a) Since the expectation of a sum equals the sum of the expectations also holds for vectors,

$$E[\mathbf{M}(n)] = \frac{1}{n}\sum_{i=1}^{n} E[\mathbf{X}(i)] = \frac{1}{n}\sum_{i=1}^{n} \mu_{\mathbf{X}} = \mu_{\mathbf{X}}. \tag{1}$$

(b) The jth component of $\mathbf{M}(n)$ is $M_j(n) = \frac{1}{n}\sum_{i=1}^{n} X_j(i)$, which is just the sample mean of X_j. Defining $A_j = \{|M_j(n) - \mu_j| \geq c\}$, we observe that

$$P\left[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c\right] = P\left[A_1 \cup A_2 \cup \cdots \cup A_k\right]. \tag{2}$$

Applying the Chebyshev inequality to $M_j(n)$, we find that

$$P\left[A_j\right] \leq \frac{\mathrm{Var}[M_j(n)]}{c^2} = \frac{\sigma_j^2}{nc^2}. \tag{3}$$

By the union bound,

$$P\left[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c\right] \leq \sum_{j=1}^{k} P\left[A_j\right] \leq \frac{1}{nc^2}\sum_{j=1}^{k} \sigma_j^2 \tag{4}$$

Since $\sum_{j=1}^{k} \sigma_j^2 < \infty$, $\lim_{n\to\infty} P[\max_{j=1,\ldots,k} |M_j(n) - \mu_j| \geq c] = 0$.

Problem 7.3.6 Solution

(a) From Theorem 6.2, we have

$$\operatorname{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \operatorname{Var}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \operatorname{Cov}[X_i, X_j] \tag{1}$$

Note that $\operatorname{Var}[X_i] = \sigma^2$ and for $j > i$, $\operatorname{Cov}[X_i, X_j] = \sigma^2 a^{j-i}$. This implies

$$\begin{aligned}\operatorname{Var}[X_1 + \cdots + X_n] &= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a^{j-i} &(2)\\ &= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} \left(a + a^2 + \cdots + a^{n-i}\right) &(3)\\ &= n\sigma^2 + \frac{2a\sigma^2}{1-a}\sum_{i=1}^{n-1}(1 - a^{n-i}) &(4)\end{aligned}$$

With some more algebra, we obtain

$$\begin{aligned}\operatorname{Var}[X_1 + \cdots + X_n] &= n\sigma^2 + \frac{2a\sigma^2}{1-a}(n-1) - \frac{2a\sigma^2}{1-a}\left(a + a^2 + \cdots + a^{n-1}\right) &(5)\\ &= \left(\frac{n(1+a)\sigma^2}{1-a}\right) - \frac{2a\sigma^2}{1-a} - 2\sigma^2\left(\frac{a}{1-a}\right)^2 (1 - a^{n-1}) &(6)\end{aligned}$$

Since $a/(1-a)$ and $1 - a^{n-1}$ are both nonnegative,

$$\operatorname{Var}[X_1 + \cdots + X_n] \le n\sigma^2\left(\frac{1+a}{1-a}\right) \tag{7}$$

(b) Since the expected value of a sum equals the sum of the expected values,

$$E[M(X_1, \ldots, X_n)] = \frac{E[X_1] + \cdots + E[X_n]}{n} = \mu \tag{8}$$

The variance of $M(X_1, \ldots, X_n)$ is

$$\operatorname{Var}[M(X_1, \ldots, X_n)] = \frac{\operatorname{Var}[X_1 + \cdots + X_n]}{n^2} \le \frac{\sigma^2(1+a)}{n(1-a)} \tag{9}$$

Applying the Chebyshev inequality to $M(X_1, \ldots, X_n)$ yields

$$P[|M(X_1, \ldots, X_n) - \mu| \ge c] \le \frac{\operatorname{Var}[M(X_1, \ldots, X_n)]}{c^2} \le \frac{\sigma^2(1+a)}{n(1-a)c^2} \tag{10}$$

(c) Taking the limit as n approaches infinity of the bound derived in part (b) yields

$$\lim_{n\to\infty} P[|M(X_1, \ldots, X_n) - \mu| \ge c] \le \lim_{n\to\infty} \frac{\sigma^2(1+a)}{n(1-a)c^2} = 0 \tag{11}$$

Thus

$$\lim_{n\to\infty} P[|M(X_1, \ldots, X_n) - \mu| \ge c] = 0 \tag{12}$$

Problem 7.4.1 Solution

$$P_X(x) = \begin{cases} 0.1 & x = 0 \\ 0.9 & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) $E[X]$ is in fact the same as $P_X(1)$ because X is a Bernoulli random variable.

(b) We can use the Chebyshev inequality to find

$$P[|M_{90}(X) - P_X(1)| \geq .05] = P[|M_{90}(X) - E[X]| \geq .05] \leq \alpha \tag{2}$$

In particular, the Chebyshev inequality states that

$$\alpha = \frac{\sigma_X^2}{90(.05)^2} = \frac{.09}{90(.05)^2} = 0.4 \tag{3}$$

(c) Now we wish to find the value of n such that $P[|M_n(X) - P_X(1)| \geq .03] \leq .01$. From the Chebyshev inequality, we write

$$0.1 = \frac{\sigma_X^2}{n(.03)^2}. \tag{4}$$

Since $\sigma_X^2 = 0.09$, solving for n yields $n = 100$.

Problem 7.4.3 Solution

(a) Since X_A is a Bernoulli ($p = P[A]$) random variable,

$$E[X_A] = P[A] = 0.8, \qquad \operatorname{Var}[X_A] = P[A](1 - P[A]) = 0.16. \tag{1}$$

(b) Let $X_{A,i}$ to denote X_A on the ith trial. Since $\hat{P}_n(A) = M_n(X_A) = \frac{1}{n}\sum_{i=1}^{n} X_{A,i}$,

$$\operatorname{Var}[\hat{P}_n(A)] = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{Var}[X_{A,i}] = \frac{P[A](1 - P[A])}{n}. \tag{2}$$

(c) Since $\hat{P}_{100}(A) = M_{100}(X_A)$, we can use Theorem 7.12(b) to write

$$P\left[\left|\hat{P}_{100}(A) - P[A]\right| < c\right] \geq 1 - \frac{\operatorname{Var}[X_A]}{100c^2} = 1 - \frac{0.16}{100c^2} = 1 - \alpha. \tag{3}$$

For $c = 0.1$, $\alpha = 0.16/[100(0.1)^2] = 0.16$. Thus, with 100 samples, our confidence coefficient is $1 - \alpha = 0.84$.

(d) In this case, the number of samples n is unknown. Once again, we use Theorem 7.12(b) to write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| < c\right] \geq 1 - \frac{\operatorname{Var}[X_A]}{nc^2} = 1 - \frac{0.16}{nc^2} = 1 - \alpha. \tag{4}$$

For $c = 0.1$, we have confidence coefficient $1 - \alpha = 0.95$ if $\alpha = 0.16/[n(0.1)^2] = 0.05$, or $n = 320$.

Problem 7.4.4 Solution
Since $E[X] = \mu_X = p$ and $\text{Var}[X] = p(1-p)$, we use Theorem 7.12(b) to write

$$P\left[|M_{100}(X) - p| < c\right] \geq 1 - \frac{p(1-p)}{100c^2} = 1 - \alpha. \quad (1)$$

For confidence coefficient 0.99, we require

$$\frac{p(1-p)}{100c^2} \leq 0.01 \quad \text{or} \quad c \geq \sqrt{p(1-p)}. \quad (2)$$

Since p is unknown, we must ensure that the constraint is met for every value of p. The worst case occurs at $p = 1/2$ which maximizes $p(1-p)$. In this case, $c = \sqrt{1/4} = 1/2$ is the smallest value of c for which we have confidence coefficient of at least 0.99.

If $M_{100}(X) = 0.06$, our interval estimate for p is

$$M_{100}(X) - c < p < M_{100}(X) + c. \quad (3)$$

Since $p \geq 0$, $M_{100}(X) = 0.06$ and $c = 0.5$ imply that our interval estimate is

$$0 \leq p < 0.56. \quad (4)$$

Our interval estimate is not very tight because because 100 samples is not very large for a confidence coefficient of 0.99.

Problem 7.4.6 Solution
Both questions can be answered using the following equation from Example 7.6:

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{P[A](1 - P[A])}{nc^2} \quad (1)$$

The unusual part of this problem is that we are given the true value of $P[A]$. Since $P[A] = 0.01$, we can write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{0.0099}{nc^2} \quad (2)$$

(a) In this part, we meet the requirement by choosing $c = 0.001$ yielding

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq 0.001\right] \leq \frac{9900}{n} \quad (3)$$

Thus to have confidence level 0.01, we require that $9900/n \leq 0.01$. This requires $n \geq 990{,}000$.

(b) In this case, we meet the requirement by choosing $c = 10^{-3}P[A] = 10^{-5}$. This implies

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \geq c\right] \leq \frac{P[A](1 - P[A])}{nc^2} = \frac{0.0099}{n10^{-10}} = \frac{9.9 \times 10^7}{n} \quad (4)$$

The confidence level 0.01 is met if $9.9 \times 10^7/n = 0.01$ or $n = 9.9 \times 10^9$.