Problem 6.1.5 Solution

This problem should be in either Chapter 10 or Chapter 11.

Since each X_i has zero mean, the mean of Y_n is

$$E[Y_n] = E[X_n + X_{n-1} + X_{n-2}]/3 = 0 \tag{1}$$

Since Y_n has zero mean, the variance of Y_n is

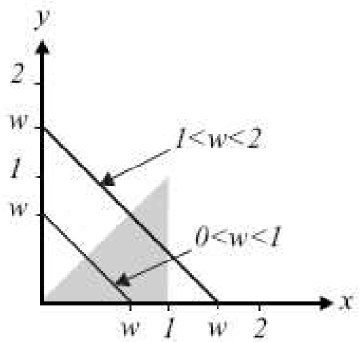
$$\begin{aligned}
\text{Var}[Y_n] &= E\left[Y_n^2\right] &(2)\\
&= \frac{1}{9}E\left[(X_n + X_{n-1} + X_{n-2})^2\right] &(3)\\
&= \frac{1}{9}E\left[X_n^2 + X_{n-1}^2 + X_{n-2}^2 + 2X_nX_{n-1} + 2X_nX_{n-2} + 2X_{n-1}X_{n-2}\right] &(4)\\
&= \frac{1}{9}(1+1+1+2/4+0+2/4) = \frac{4}{9} &(5)
\end{aligned}$$

Problem 6.2.4 Solution

In this problem, X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the PDF of W using Theorem 6.4: $f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\,dx$. The only tricky part remaining is to determine the limits of the integration. First, for $w < 0$, $f_W(w) = 0$. The two remaining cases are shown in the accompanying figure. The shaded area shows where the joint PDF $f_{X,Y}(x,y)$ is nonzero. The diagonal lines depict $y = w - x$ as a function of x. The intersection of the diagonal line and the shaded area define our limits of integration.



For $0 \le w \le 1$,

$$\begin{aligned}
f_W(w) &= \int_{w/2}^{w} 8x(w-x)\,dx &(2)\\
&= 4wx^2 - 8x^3/3\Big|_{w/2}^{w} = 2w^3/3 &(3)
\end{aligned}$$

For $1 \le w \le 2$,

$$\begin{aligned}
f_W(w) &= \int_{w/2}^{1} 8x(w-x)\,dx &(4)\\
&= 4wx^2 - 8x^3/3\Big|_{w/2}^{1} &(5)\\
&= 4w - 8/3 - 2w^3/3 &(6)
\end{aligned}$$

Since $X + Y \le 2$, $f_W(w) = 0$ for $w > 2$. Hence the complete expression for the PDF of W is

$$f_W(w) = \begin{cases} 2w^3/3 & 0 \le w \le 1 \\ 4w - 8/3 - 2w^3/3 & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 6.2.5 Solution
We first find the CDF of W following the same procedure as in the proof of Theorem 6.4.

$$F_W(w) = P[X \leq Y + w] = \int_{-\infty}^{\infty} \int_{-\infty}^{y+w} f_{X,Y}(x,y)\, dx\, dy \tag{1}$$

By taking the derivative with respect to w, we obtain

$$f_W(w) = \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} \frac{d}{dw}\left(\int_{-\infty}^{y+w} f_{X,Y}(x,y)\, dx\right) dy \tag{2}$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(w+y,y)\, dy \tag{3}$$

With the variable substitution $y = x - w$, we have $dy = dx$ and

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, x-w)\, dx \tag{4}$$

Problem 6.3.5 Solution
The PMF of K is

$$P_K(k) = \begin{cases} 1/n & k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The corresponding MGF of K is

$$\phi_K(s) = E\left[e^{sK}\right] = \frac{1}{n}\left(e^s + e^{2s} + \cdots + e^{ns}\right) \tag{2}$$

$$= \frac{e^s}{n}\left(1 + e^s + e^{2s} + \cdots + e^{(n-1)s}\right) \tag{3}$$

$$= \frac{e^s(e^{ns}-1)}{n(e^s-1)} \tag{4}$$

We can evaluate the moments of K by taking derivatives of the MGF. Some algebra will show that

$$\frac{d\phi_K(s)}{ds} = \frac{ne^{(n+2)s} - (n+1)e^{(n+1)s} + e^s}{n(e^s-1)^2} \tag{5}$$

Evaluating $d\phi_K(s)/ds$ at $s = 0$ yields $0/0$. Hence, we apply l'Hôpital's rule twice (by twice differentiating the numerator and twice differentiating the denominator) when we write

$$\left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = \lim_{s\to 0} \frac{n(n+2)e^{(n+2)s} - (n+1)^2 e^{(n+1)s} + e^s}{2n(e^s-1)} \tag{6}$$

$$= \lim_{s\to 0} \frac{n(n+2)^2 e^{(n+2)s} - (n+1)^3 e^{(n+1)s} + e^s}{2ne^s} = (n+1)/2 \tag{7}$$

A significant amount of algebra will show that the second derivative of the MGF is

$$\frac{d^2\phi_K(s)}{ds^2} = \frac{n^2 e^{(n+3)s} - (2n^2+2n-1)e^{(n+2)s} + (n+1)^2 e^{(n+1)s} - e^{2s} - e^s}{n(e^s-1)^3} \tag{8}$$

Evaluating $d^2\phi_K(s)/ds^2$ at $s = 0$ yields $0/0$. Because $(e^s - 1)^3$ appears in the denominator, we need to use l'Hôpital's rule three times to obtain our answer.

$$\left.\frac{d^2\phi_K(s)}{ds^2}\right|_{s=0} = \lim_{s\to 0} \frac{n^2(n+3)^3 e^{(n+3)s} - (2n^2+2n-1)(n+2)^3 e^{(n+2)s} + (n+1)^5 - 8e^{2s} - e^s}{6ne^s} \tag{9}$$

$$= \frac{n^2(n+3)^3 - (2n^2+2n-1)(n+2)^3 + (n+1)^5 - 9}{6n} \tag{10}$$

$$= (2n+1)(n+1)/6 \tag{11}$$

We can use these results to derive two well known results. We observe that we can directly use the PMF $P_K(k)$ to calculate the moments

$$E[K] = \frac{1}{n}\sum_{k=1}^{n} k \qquad E\left[K^2\right] = \frac{1}{n}\sum_{k=1}^{n} k^2 \tag{12}$$

Using the answers we found for $E[K]$ and $E[K^2]$, we have the formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{13}$$

Problem 6.4.7 Solution

By Theorem 6.8, we know that $\phi_M(s) = [\phi_K(s)]^n$.

(a) The first derivative of $\phi_M(s)$ is

$$\frac{d\phi_M(s)}{ds} = n\left[\phi_K(s)\right]^{n-1}\frac{d\phi_K(s)}{ds} \tag{1}$$

We can evaluate $d\phi_M(s)/ds$ at $s = 0$ to find $E[M]$.

$$E[M] = \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} = n\left[\phi_K(s)\right]^{n-1}\left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = nE[K] \tag{2}$$

(b) The second derivative of $\phi_M(s)$ is

$$\frac{d^2\phi_M(s)}{ds^2} = n(n-1)\left[\phi_K(s)\right]^{n-2}\left(\frac{d\phi_K(s)}{ds}\right)^2 + n\left[\phi_K(s)\right]^{n-1}\frac{d^2\phi_K(s)}{ds^2} \tag{3}$$

Evaluating the second derivative at $s = 0$ yields

$$E\left[M^2\right] = \left.\frac{d^2\phi_M(s)}{ds^2}\right|_{s=0} = n(n-1)\left(E[K]\right)^2 + nE\left[K^2\right] \tag{4}$$

Problem 6.5.2 Solution

The number N of passes thrown has the Poisson PMF and MGF

$$P_N(n) = \begin{cases} (30)^n e^{-30}/n! & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N(s) = e^{30(e^s - 1)} \tag{1}$$

Let $X_i = 1$ if pass i is thrown and completed and otherwise $X_i = 0$. The PMF and MGF of each X_i is

$$P_{X_i}(x) = \begin{cases} 1/3 & x = 0 \\ 2/3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_{X_i}(s) = 1/3 + (2/3)e^s \tag{2}$$

The number of completed passes can be written as the random sum of random variables

$$K = X_1 + \cdots + X_N \tag{3}$$

Since each X_i is independent of N, we can use Theorem 6.12 to write

$$\phi_K(s) = \phi_N(\ln \phi_X(s)) = e^{30(\phi_X(s) - 1)} = e^{30(2/3)(e^s - 1)} \tag{4}$$

We see that K has the MGF of a Poisson random variable with mean $E[K] = 30(2/3) = 20$, variance $\text{Var}[K] = 20$, and PMF

$$P_K(k) = \begin{cases} (20)^k e^{-20}/k! & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Problem 6.5.8 Solution

Using N to denote the number of games played, we can write the total number of points earned as the random sum

$$Y = X_1 + X_2 + \cdots + X_N \tag{1}$$

(a) It is tempting to use Theorem 6.12 to find $\phi_Y(s)$; however, this would be wrong since each X_i is not independent of N. In this problem, we must start from first principles using iterated expectations.

$$\phi_Y(s) = E\left[E\left[e^{s(X_1 + \cdots + X_N)}|N\right]\right] = \sum_{n=1}^{\infty} P_N(n)\, E\left[e^{s(X_1 + \cdots + X_n)}|N = n\right] \tag{2}$$

Given $N = n$, $X_1, \ldots, X_n$ are independent so that

$$E\left[e^{s(X_1 + \cdots + X_n)}|N = n\right] = E\left[e^{sX_1}|N = n\right] E\left[e^{sX_2}|N = n\right] \cdots E\left[e^{sX_n}|N = n\right] \tag{3}$$

Given $N = n$, we know that games 1 through $n - 1$ were either wins or ties and that game n was a loss. That is, given $N = n$, $X_n = 0$ and for $i < n$, $X_i \neq 0$. Moreover, for $i < n$, X_i has the conditional PMF

$$P_{X_i|N=n}(x) = P_{X_i|X_i \neq 0}(x) = \begin{cases} 1/2 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

These facts imply

$$E\left[e^{sX_n}|N = n\right] = e^0 = 1 \tag{5}$$

and that for $i < n$,

$$E\left[e^{sX_i}|N = n\right] = (1/2)e^s + (1/2)e^{2s} = e^s/2 + e^{2s}/2 \tag{6}$$

Now we can find the MGF of Y.

$$\phi_Y(s) = \sum_{n=1}^{\infty} P_N(n)\, E\left[e^{sX_1}|N=n\right] E\left[e^{sX_2}|N=n\right] \cdots E\left[e^{sX_n}|N=n\right] \tag{7}$$

$$= \sum_{n=1}^{\infty} P_N(n) \left[e^s/2 + e^{2s}/2\right]^{n-1} = \frac{1}{e^s/2+e^{2s}/2} \sum_{n=1}^{\infty} P_N(n) \left[e^s/2 + e^{2s}/2\right]^{n} \tag{8}$$

It follows that

$$\phi_Y(s) = \frac{1}{e^s/2+e^{2s}/2} \sum_{n=1}^{\infty} P_N(n)\, e^{n \ln[(e^s+e^{2s})/2]} = \frac{\phi_N(\ln[e^s/2+e^{2s}/2])}{e^s/2+e^{2s}/2} \tag{9}$$

The tournament ends as soon as you lose a game. Since each game is a loss with probability 1/3 independent of any previous game, the number of games played has the geometric PMF and corresponding MGF

$$P_N(n) = \begin{cases} (2/3)^{n-1}(1/3) & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N(s) = \frac{(1/3)e^s}{1-(2/3)e^s} \tag{10}$$

Thus, the MGF of Y is

$$\phi_Y(s) = \frac{1/3}{1-(e^s+e^{2s})/3} \tag{11}$$

(b) To find the moments of Y, we evaluate the derivatives of the MGF $\phi_Y(s)$. Since

$$\frac{d\phi_Y(s)}{ds} = \frac{e^s + 2e^{2s}}{9\left[1 - e^s/3 - e^{2s}/3\right]^2} \tag{12}$$

we see that

$$E[Y] = \left.\frac{d\phi_Y(s)}{ds}\right|_{s=0} = \frac{3}{9(1/3)^2} = 3 \tag{13}$$

If you're curious, you may notice that $E[Y] = 3$ precisely equals $E[N]E[X_i]$, the answer you would get if you mistakenly assumed that N and each X_i were independent. Although this may seem like a coincidence, its actually the result of theorem known as Wald's equality.

The second derivative of the MGF is

$$\frac{d^2\phi_Y(s)}{ds^2} = \frac{(1-e^s/3-e^{2s}/3)(e^s+4e^{2s}) + 2(e^s+2e^{2s})^2/3}{9(1-e^s/3-e^{2s}/3)^3} \tag{14}$$

The second moment of Y is

$$E\left[Y^2\right] = \left.\frac{d^2\phi_Y(s)}{ds^2}\right|_{s=0} = \frac{5/3+6}{1/3} = 23 \tag{15}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23 - 9 = 14$.

Problem 6.6.2 Solution

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable D_i as the number of data calls in a single telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_i's are independent and identically distributed withe the following PMF.

$$P_D(d) = \begin{cases} 0.8 & d = 0 \\ 0.2 & d = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the above we can determine that

$$E[D] = 0.2 \qquad \text{Var}[D] = 0.2 - 0.04 = 0.16 \tag{2}$$

With these facts, we can answer the questions posed by the problem.

(a) $E[K_{100}] = 100E[D] = 20$

(b) $\text{Var}[K_{100}] = \sqrt{100\,\text{Var}[D]} = \sqrt{16} = 4$

(c) $P[K_{100} \geq 18] = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-1/2) = \Phi(1/2) = 0.6915$

(d) $P[16 \leq K_{100} \leq 24] = \Phi(\frac{24-20}{4}) - \Phi(\frac{16-20}{4}) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826$

Problem 6.7.1 Solution

In Problem 6.2.6, we learned that a sum of iid Poisson random variables is a Poisson random variable. Hence W_n is a Poisson random variable with mean $E[W_n] = nE[K] = n$. Thus W_n has variance $\text{Var}[W_n] = n$ and PMF

$$P_{W_n}(w) = \begin{cases} n^w e^{-n}/w! & w = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

All of this implies that we can exactly calculate

$$P[W_n = n] = P_{W_n}(n) = n^n e^{-n}/n! \tag{2}$$

Since we can perform the exact calculation, using a central limit theorem may seem silly; however for large n, calculating n^n or $n!$ is difficult for large n. Moreover, it's interesting to see how good the approximation is. In this case, the approximation is

$$P[W_n = n] = P[n \leq W_n \leq n] \approx \Phi\left(\frac{n + 0.5 - n}{\sqrt{n}}\right) - \Phi\left(\frac{n - 0.5 - n}{\sqrt{n}}\right) = 2\Phi\left(\frac{1}{2\sqrt{n}}\right) - 1 \tag{3}$$

The comparison of the exact calculation and the approximation are given in the following table.

$P[W_n = n]$	$n = 1$	$n = 4$	$n = 16$	$n = 64$
exact	0.3679	0.1954	0.0992	0.0498
approximate	0.3829	0.1974	0.0995	0.0498

(4)

Problem 6.8.2 Solution
For an $N[\mu, \sigma^2]$ random variable X, we can write

$$P[X \geq c] = P[(X-\mu)/\sigma \geq (c-\mu)/\sigma] = P[Z \geq (c-\mu)/\sigma] \tag{1}$$

Since Z is $N[0,1]$, we can apply the result of Problem 6.8.1 with c replaced by $(c-\mu)/\sigma$. This yields

$$P[X \geq c] = P[Z \geq (c-\mu)/\sigma] \leq e^{-(c-\mu)^2/2\sigma^2} \tag{2}$$

Problem 6.8.5 Solution
Let $W_n = X_1 + \cdots + X_n$. Since $M_n(X) = W_n/n$, we can write

$$P[M_n(X) \geq c] = P[W_n \geq nc] \tag{1}$$

Since $\phi_{W_n}(s) = (\phi_X(s))^n$, applying the Chernoff bound to W_n yields

$$P[W_n \geq nc] \leq \min_{s \geq 0} e^{-snc}\phi_{W_n}(s) = \min_{s \geq 0} \left(e^{-sc}\phi_X(s)\right)^n \tag{2}$$

For $y \geq 0$, y^n is a nondecreasing function of y. This implies that the value of s that minimizes $e^{-sc}\phi_X(s)$ also minimizes $(e^{-sc}\phi_X(s))^n$. Hence

$$P[M_n(X) \geq c] = P[W_n \geq nc] \leq \left(\min_{s \geq 0} e^{-sc}\phi_X(s)\right)^n \tag{3}$$