Problem 3.1.3 Solution

In this problem, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -5 \\ (w+5)/8 & -5 \le w < -3 \\ 1/4 & -3 \le w < 3 \\ 1/4 + 3(w-3)/8 & 3 \le w < 5 \\ 1 & w \ge 5. \end{cases} \tag{1}$$

Each question can be answered directly from this CDF.

(a)
$$P[W \le 4] = F_W(4) = 1/4 + 3/8 = 5/8. \tag{2}$$

(b)
$$P[-2 < W \le 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0. \tag{3}$$

(c)
$$P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 3/4 \tag{4}$$

(d) By inspection of $F_W(w)$, we observe that $P[W \le a] = F_W(a) = 1/2$ for a in the range $3 \le a \le 5$. In this range,
$$F_W(a) = 1/4 + 3(a-3)/8 = 1/2 \tag{5}$$
This implies $a = 11/3$.

Problem 3.2.3 Solution

We find the PDF by taking the derivative of $F_U(u)$ on each piece that $F_U(u)$ is defined. The CDF and corresponding PDF of U are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{1}$$

Problem 3.2.4 Solution

For $x < 0$, $F_X(x) = 0$. For $x \ge 0$,

$$\begin{aligned} F_X(x) &= \int_0^x f_X(y)\, dy & (1)\\ &= \int_0^x a^2 y e^{-a^2 y^2/2}\, dy & (2)\\ &= -e^{-a^2y^2/2}\Big|_0^x = 1 - e^{-a^2x^2/2} & (3) \end{aligned}$$

A complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-a^2x^2/2} & x \ge 0 \end{cases} \tag{4}$$

Problem 3.3.2 Solution

(a) Since the PDF is uniform over [1,9]

$$E[X] = \frac{1+9}{2} = 5 \qquad \operatorname{Var}[X] = \frac{(9-1)^2}{12} = \frac{16}{3} \tag{1}$$

(b) Define $h(X) = 1/\sqrt{X}$ then

$$h(E[X]) = 1/\sqrt{5} \tag{2}$$

$$E[h(X)] = \int_1^9 \frac{x^{-1/2}}{8}\,dx = 1/2 \tag{3}$$

Problem 3.3.8 Solution

The Pareto (α, μ) random variable has PDF

$$f_X(x) = \begin{cases} (\alpha/\mu)\,(x/\mu)^{-(\alpha+1)} & x \geq \mu \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The nth moment is

$$E[X^n] = \int_\mu^\infty x^n \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{-(\alpha+1)} dx = \mu^n \int_\mu^\infty \frac{\alpha}{\mu}\left(\frac{x}{\mu}\right)^{-(\alpha-n+1)} dx \tag{2}$$

With the variable substitution $y = x/\mu$, we obtain

$$E[X^n] = \alpha\mu^n \int_1^\infty y^{-(\alpha-n+1)}\,dy \tag{3}$$

We see that $E[X^n] < \infty$ if and only if $\alpha - n + 1 > 1$, or, equivalently, $n < \alpha$. In this case,

$$E[X^n] = \left.\frac{\alpha\mu^n}{-(\alpha-n+1)+1} y^{-(\alpha-n+1)+1}\right|_{y=1}^{y=\infty} \tag{4}$$

$$= \left.\frac{-\alpha\mu^n}{\alpha-n} y^{-(\alpha-n)}\right|_{y=1}^{y=\infty} = \frac{\alpha\mu^n}{\alpha-n} \tag{5}$$

Problem 3.4.1 Solution

The reflected power Y has an exponential $(\lambda = 1/P_0)$ PDF. From Theorem 3.8, $E[Y] = P_0$. The probability that an aircraft is correctly identified is

$$P[Y > P_0] = \int_{P_0}^\infty \frac{1}{P_0} e^{-y/P_0}\,dy = e^{-1}. \tag{1}$$

Fortunately, real radar systems offer better performance.

Problem 3.4.9 Solution

Let X denote the holding time of a call. The PDF of X is

$$f_X(x) = \begin{cases} (1/\tau)e^{-x/\tau} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We will use $C_A(X)$ and $C_B(X)$ to denote the cost of a call under the two plans. From the problem statement, we note that $C_A(X) = 10X$ so that $E[C_A(X)] = 10E[X] = 10\tau$. On the other hand

$$C_B(X) = 99 + 10(X - 20)^+ \tag{2}$$

where $y^+ = y$ if $y \geq 0$; otherwise $y^+ = 0$ for $y < 0$. Thus,

$$\begin{aligned} E[C_B(X)] &= E\left[99 + 10(X-20)^+\right] && (3)\\ &= 99 + 10E\left[(X-20)^+\right] && (4)\\ &= 99 + 10E\left[(X-20)^+ | X \leq 20\right] P[X \leq 20] \\ &\qquad + 10E\left[(X-20)^+ | X > 20\right] P[X > 20] && (5) \end{aligned}$$

Given $X \leq 20$, $(X-20)^+ = 0$. Thus $E[(X-20)^+ | X \leq 20] = 0$ and

$$E[C_B(X)] = 99 + 10E[(X-20)|X > 20]\, P[X > 20] \tag{6}$$

Finally, we observe that $P[X > 20] = e^{-20/\tau}$ and that

$$E[(X-20)|X > 20] = \tau \tag{7}$$

since given $X \geq 20$, $X - 20$ has a PDF identical to X by the memoryless property of the exponential random variable. Thus,

$$E[C_B(X)] = 99 + 10\tau e^{-20/\tau} \tag{8}$$

Some numeric comparisons show that $E[C_B(X)] \leq E[C_A(X)]$ if $\tau > 12.34$ minutes. That is, the flat price for the first 20 minutes is a good deal only if your average phone call is sufficiently long.

Problem 3.4.11 Solution

For an Erlang (n, λ) random variable X, the kth moment is

$$\begin{aligned} E\left[X^k\right] &= \int_0^\infty x^k f_X(x)\, dt && (1)\\ &= \int_0^\infty \frac{\lambda^n x^{n+k-1}}{(n-1)!} e^{-\lambda x}\, dt = \frac{(n+k-1)!}{\lambda^k (n-1)!} \underbrace{\int_0^\infty \frac{\lambda^{n+k} x^{n+k-1}}{(n+k-1)!} e^{-\lambda t}\, dt}_{1} && (2) \end{aligned}$$

The above marked integral equals 1 since it is the integral of an Erlang PDF with parameters λ and $n + k$ over all possible values. Hence,

$$E\left[X^k\right] = \frac{(n+k-1)!}{\lambda^k (n-1)!} \tag{3}$$

This implies that the first and second moments are

$$E[X] = \frac{n!}{(n-1)!\lambda} = \frac{n}{\lambda} \qquad E\left[X^2\right] = \frac{(n+1)!}{\lambda^2 (n-1)!} = \frac{(n+1)n}{\lambda^2} \tag{4}$$

It follows that the variance of X is n/λ^2.

Problem 3.4.14 Solution

(a) Since $f_X(x) \geq 0$ and $x \geq r$ over the entire integral, we can write

$$\int_r^\infty x f_X(x)\, dx \geq \int_r^\infty r f_X(x)\, dx = r P[X > r] \tag{1}$$

(b) We can write the expected value of X in the form

$$E[X] = \int_0^r x f_X(x)\, dx + \int_r^\infty x f_X(x)\, dx \tag{2}$$

Hence,

$$r P[X > r] \leq \int_r^\infty x f_X(x)\, dx = E[X] - \int_0^r x f_X(x)\, dx \tag{3}$$

Allowing r to approach infinity yields

$$\lim_{r\to\infty} r P[X > r] \leq E[X] - \lim_{r\to\infty} \int_0^r x f_X(x)\, dx = E[X] - E[X] = 0 \tag{4}$$

Since $rP[X > r] \geq 0$ for all $r \geq 0$, we must have $\lim_{r\to\infty} rP[X > r] = 0$.

(c) We can use the integration by parts formula $\int u\, dv = uv - \int v\, du$ by defining $u = 1 - F_X(x)$ and $dv = dx$. This yields

$$\int_0^\infty [1 - F_X(x)]\, dx = x[1 - F_X(x)]|_0^\infty + \int_0^\infty x f_X(x)\, dx \tag{5}$$

By applying part (a), we now observe that

$$x[1 - F_X(x)]|_0^\infty = \lim_{r\to\infty} r[1 - F_X(r)] - 0 = \lim_{r\to\infty} r P[X > r] \tag{6}$$

By part (b), $\lim_{r\to\infty} rP[X > r] = 0$ and this implies $x[1 - F_X(x)]|_0^\infty = 0$. Thus,

$$\int_0^\infty [1 - F_X(x)]\, dx = \int_0^\infty x f_X(x)\, dx = E[X] \tag{7}$$

Problem 3.5.4 Solution

Repeating Definition 3.11,

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2}\, du \tag{1}$$

Making the substitution $x = u/\sqrt{2}$, we have

$$Q(z) = \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{2}}^\infty e^{-x^2}\, dx = \frac{1}{2}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right) \tag{2}$$

Problem 3.5.5 Solution

Moving to Antarctica, we find that the temperature, T is still Gaussian but with variance 225. We also know that with probability 1/2, T exceeds 10 degrees. First we would like to find the mean temperature, and we do so by looking at the second fact.

$$P[T > 10] = 1 - P[T \leq 10] = 1 - \Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{1}$$

By looking at the table we find that if $\Phi(\Gamma) = 1/2$, then $\Gamma = 0$. Therefore,

$$\Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{2}$$

implies that $(10 - \mu_T)/15 = 0$ or $\mu_T = 10$. Now we have a Gaussian T with mean 10 and standard deviation 15. So we are prepared to answer the following problems.

$$P[T > 32] = 1 - P[T \leq 32] = 1 - \Phi\left(\frac{32-10}{15}\right) \tag{3}$$
$$= 1 - \Phi(1.45) = 1 - 0.926 = 0.074 \tag{4}$$
$$P[T < 0] = F_T(0) = \Phi\left(\frac{0-10}{15}\right) \tag{5}$$
$$= \Phi(-2/3) = 1 - \Phi(2/3) \tag{6}$$
$$= 1 - \Phi(0.67) = 1 - 0.749 = 0.251 \tag{7}$$
$$P[T > 60] = 1 - P[T \leq 60] = 1 - F_T(60) \tag{8}$$
$$= 1 - \Phi\left(\frac{60-10}{15}\right) = 1 - \Phi(10/3) \tag{9}$$
$$= Q(3.33) = 4.34 \cdot 10^{-4} \tag{10}$$

Problem 3.5.10 Solution

This problem is mostly calculus and only a little probability. From the problem statement, the SNR Y is an exponential $(1/\gamma)$ random variable with PDF

$$f_Y(y) = \begin{cases} (1/\gamma)e^{-y/\gamma} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, from the problem statement, the BER is

$$\overline{P}_e = E[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy = \int_0^{\infty} Q(\sqrt{2y}) \frac{y}{\gamma} e^{-y/\gamma}\, dy \tag{2}$$

Like most integrals with exponential factors, its a good idea to try integration by parts. Before doing so, we recall that if X is a Gaussian $(0,1)$ random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x). \tag{3}$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{dQ(x)}{dx} = -\frac{dF_X(x)}{dx} = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

To solve the integral, we use the integration by parts formula $\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du$, where

$$u = Q(\sqrt{2y}) \qquad dv = \frac{1}{\gamma} e^{-y/\gamma}\, dy \tag{5}$$
$$du = Q'(\sqrt{2y}) \frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi y}} \qquad v = -e^{-y/\gamma} \tag{6}$$

From integration by parts, it follows that

$$\overline{P}_e = uv|_0^{\infty} - \int_0^{\infty} v\, du = -Q(\sqrt{2y}) e^{-y/\gamma} \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-y[1+(1/\gamma)]}\, dy \tag{7}$$
$$= 0 + Q(0)e^{-0} - \frac{1}{2\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y/\bar{\gamma}}\, dy \tag{8}$$

where $\bar{\gamma} = \gamma/(1+\gamma)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\bar{\gamma}$, we obtain

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \int_0^{\infty} t^{-1/2} e^{-t}\, dt \tag{9}$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \Gamma(1/2) = \frac{1}{2}\left[1 - \sqrt{\bar{\gamma}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma}{1+\gamma}}\right] \tag{10}$$

Problem 3.6.5 Solution

The PMF of a geometric random variable with mean $1/p$ is

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The corresponding PDF is

$$\begin{aligned} f_X(x) &= p\delta(x-1) + p(1-p)\delta(x-2) + \cdots && (2) \\ &= \sum_{j=1}^{\infty} p(1-p)^{j-1}\delta(x-j) && (3) \end{aligned}$$

Problem 3.6.6 Solution

(a) Since the conversation time cannot be negative, we know that $F_W(w) = 0$ for $w < 0$. The conversation time W is zero iff either the phone is busy, no one answers, or if the conversation time X of a completed call is zero. Let A be the event that the call is answered. Note that the event A^c implies $W = 0$. For $w \geq 0$,

$$F_W(w) = P[A^c] + P[A] F_{W|A}(w) = (1/2) + (1/2)F_X(w) \tag{1}$$

Thus the complete CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + (1/2)F_X(w) & w \geq 0 \end{cases} \tag{2}$$

(b) By taking the derivative of $F_W(w)$, the PDF of W is

$$f_W(w) = \begin{cases} (1/2)\delta(w) + (1/2)f_X(w) & \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Next, we keep in mind that since X must be nonnegative, $f_X(x) = 0$ for $x < 0$. Hence,

$$f_W(w) = (1/2)\delta(w) + (1/2)f_X(w) \tag{4}$$

(c) From the PDF $f_W(w)$, calculating the moments is straightforward.

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} w f_X(w)\, dw = E[X]/2 \tag{5}$$

The second moment is

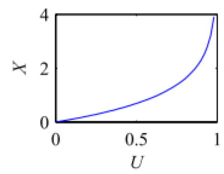
$$E\left[W^2\right] = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} w^2 f_X(w)\, dw = E\left[X^2\right]/2 \tag{6}$$

The variance of W is

$$\begin{aligned} \mathrm{Var}[W] = E\left[W^2\right] - (E[W])^2 &= E\left[X^2\right]/2 - (E[X]/2)^2 && (7) \\ &= (1/2)\,\mathrm{Var}[X] + (E[X])^2/4 && (8) \end{aligned}$$

Problem 3.7.5 Solution

Before solving for the PDF, it is helpful to have a sketch of the function $X = -\ln(1-U)$.



(a) From the sketch, we observe that X will be nonnegative. Hence $F_X(x) = 0$ for $x < 0$. Since U has a uniform distribution on $[0,1]$, for $0 \le u \le 1$, $P[U \le u] = u$. We use this fact to find the CDF of X. For $x \ge 0$,

$$F_X(x) = P[-\ln(1-U) \le x] = P\left[1 - U \ge e^{-x}\right] = P\left[U \le 1 - e^{-x}\right] \tag{1}$$

For $x \ge 0$, $0 \le 1 - e^{-x} \le 1$ and so

$$F_X(x) = F_U\left(1 - e^{-x}\right) = 1 - e^{-x} \tag{2}$$

The complete CDF can be written as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \ge 0 \end{cases} \tag{3}$$

(b) By taking the derivative, the PDF is

$$f_X(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Thus, X has an exponential PDF. In fact, since most computer languages provide uniform $[0,1]$ random numbers, the procedure outlined in this problem provides a way to generate exponential random variables from uniform random variables.

(c) Since X is an exponential random variable with parameter $a = 1$, $E[X] = 1$.

Problem 3.7.8 Solution

Let X denote the position of the pointer and Y denote the area within the arc defined by the stopping position of the pointer.

(a) If the disc has radius r, then the area of the disc is πr^2. Since the circumference of the disc is 1 and X is measured around the circumference, $Y = \pi r^2 X$. For example, when $X = 1$, the shaded area is the whole disc and $Y = \pi r^2$. Similarly, if $X = 1/2$, then $Y = \pi r^2/2$ is half the area of the disc. Since the disc has circumference 1, $r = 1/(2\pi)$ and

$$Y = \pi r^2 X = \frac{X}{4\pi} \tag{1}$$

(b) The CDF of Y can be expressed as

$$F_Y(y) = P[Y \le y] = P\left[\frac{X}{4\pi} \le y\right] = P[X \le 4\pi y] = F_X(4\pi y) \tag{2}$$

Therefore the CDF is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 4\pi y & 0 \le y \le \frac{1}{4\pi} \\ 1 & y \ge \frac{1}{4\pi} \end{cases} \tag{3}$$

(c) By taking the derivative of the CDF, the PDF of Y is

$$f_Y(y) = \begin{cases} 4\pi & 0 \le y \le \frac{1}{4\pi} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The expected value of Y is $E[Y] = \int_0^{1/(4\pi)} 4\pi y \, dy = 1/(8\pi)$.

Problem 3.7.13 Solution

If X has a uniform distribution from 0 to 1 then the PDF and corresponding CDF of X are

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1}$$

For $b - a > 0$, we can find the CDF of the function $Y = a + (b-a)X$

$$\begin{aligned} F_Y(y) = P[Y \le y] &= P[a + (b-a)X \le y] && (2) \\ &= P\left[X \le \frac{y-a}{b-a}\right] && (3) \\ &= F_X\left(\frac{y-a}{b-a}\right) = \frac{y-a}{b-a} && (4) \end{aligned}$$

Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \le y \le b \\ 1 & y \ge b \end{cases} \tag{5}$$

By differentiating with respect to y we arrive at the PDF

$$f_Y(y) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

which we recognize as the PDF of a uniform (a, b) random variable.

Problem 3.7.17 Solution

Understanding this claim may be harder than completing the proof. Since $0 \le F(x) \le 1$, we know that $0 \le U \le 1$. This implies $F_U(u) = 0$ for $u < 0$ and $F_U(u) = 1$ for $u \ge 1$. Moreover, since $F(x)$ is an increasing function, we can write for $0 \le u \le 1$,

$$F_U(u) = P[F(X) \le u] = P\left[X \le F^{-1}(u)\right] = F_X\left(F^{-1}(u)\right) \tag{1}$$

Since $F_X(x) = F(x)$, we have for $0 \le u \le 1$,

$$F_U(u) = F(F^{-1}(u)) = u \tag{2}$$

Hence the complete CDF of U is

$$F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \le u < 1 \\ 1 & u \ge 1 \end{cases} \tag{3}$$

That is, U is a uniform $[0, 1]$ random variable.

Problem 3.8.2 Solution
From Definition 3.6, the PDF of Y is

$$f_Y(y) = \begin{cases} (1/5)e^{-y/5} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The event A has probability

$$P[A] = P[Y < 2] = \int_0^2 (1/5)e^{-y/5}\,dy = -e^{-y/5}\Big|_0^2 = 1 - e^{-2/5} \tag{2}$$

From Definition 3.15, the conditional PDF of Y given A is

$$f_{Y|A}(y) = \begin{cases} f_Y(y)/P[A] & x \in A \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$= \begin{cases} (1/5)e^{-y/5}/(1 - e^{-2/5}) & 0 \leq y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) The conditional expected value of Y given A is

$$E[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y)\,dy = \frac{1/5}{1 - e^{-2/5}} \int_0^2 y e^{-y/5}\,dy \tag{5}$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$ with $u = y$ and $dv = e^{-y/5}\,dy$ yields

$$E[Y|A] = \frac{1/5}{1 - e^{-2/5}} \left(-5ye^{-y/5}\Big|_0^2 + \int_0^2 5e^{-y/5}\,dy \right) \tag{6}$$

$$= \frac{1/5}{1 - e^{-2/5}} \left(-10e^{-2/5} - 25e^{-y/5}\Big|_0^2 \right) \tag{7}$$

$$= \frac{5 - 7e^{-2/5}}{1 - e^{-2/5}} \tag{8}$$

Problem 3.8.7 Solution

(a) Given that a person is healthy, X is a Gaussian ($\mu = 90, \sigma = 20$) random variable. Thus,

$$f_{X|H}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{20\sqrt{2\pi}} e^{-(x-90)^2/800} \tag{1}$$

(b) Given the event H, we use the conditional PDF $f_{X|H}(x)$ to calculate the required probabilities

$$P\left[T^+|H\right] = P\left[X \geq 140|H\right] = P\left[X - 90 \geq 50|H\right] \tag{2}$$

$$= P\left[\frac{X-90}{20} \geq 2.5|H\right] = 1 - \Phi(2.5) = 0.006 \tag{3}$$

Similarly,

$$P\left[T^-|H\right] = P\left[X \leq 110|H\right] = P\left[X - 90 \leq 20|H\right] \tag{4}$$

$$= P\left[\frac{X-90}{20} \leq 1|H\right] = \Phi(1) = 0.841 \tag{5}$$

(c) Using Bayes Theorem, we have

$$P\left[H|T^-\right] = \frac{P\left[T^-|H\right]P\left[H\right]}{P\left[T^-\right]} = \frac{P\left[T^-|H\right]P\left[H\right]}{P\left[T^-|D\right]P\left[D\right] + P\left[T^-|H\right]P\left[H\right]} \tag{6}$$

In the denominator, we need to calculate

$$P\left[T^-|D\right] = P\left[X \leq 110|D\right] = P\left[X - 160 \leq -50|D\right] \tag{7}$$

$$= P\left[\frac{X-160}{40} \leq -1.25|D\right] \tag{8}$$

$$= \Phi(-1.25) = 1 - \Phi(1.25) = 0.106 \tag{9}$$

Thus,

$$P\left[H|T^-\right] = \frac{P\left[T^-|H\right]P\left[H\right]}{P\left[T^-|D\right]P\left[D\right] + P\left[T^-|H\right]P\left[H\right]} \tag{10}$$

$$= \frac{0.841(0.9)}{0.106(0.1) + 0.841(0.9)} = 0.986 \tag{11}$$

(d) Since T^-, T^0, and T^+ are mutually exclusive and collectively exhaustive,

$$P\left[T^0|H\right] = 1 - P\left[T^-|H\right] - P\left[T^+|H\right] = 1 - 0.841 - 0.006 = 0.153 \tag{12}$$

We say that a test is a failure if the result is T^0. Thus, given the event H, each test has conditional failure probability of $q = 0.153$, or success probability $p = 1 - q = 0.847$. Given H, the number of trials N until a success is a geometric (p) random variable with PMF

$$P_{N|H}(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Problem 3.8.8 Solution

(a) The event B_i that $Y = \Delta/2 + i\Delta$ occurs if and only if $i\Delta \le X < (i+1)\Delta$. In particular, since X has the uniform $(-r/2, r/2)$ PDF

$$f_X(x) = \begin{cases} 1/r & -r/2 \le x < r/2, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

we observe that

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} \frac{1}{r}\, dx = \frac{\Delta}{r} \tag{2}$$

In addition, the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} f_X(x)/P[B] & x \in B_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/\Delta & i\Delta \le x < (i+1)\Delta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

It follows that given B_i, $Z = X - Y = X - \Delta/2 - i\Delta$, which is a uniform $(-\Delta/2, \Delta/2)$ random variable. That is,

$$f_{Z|B_i}(z) = \begin{cases} 1/\Delta & -\Delta/2 \le z < \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) We observe that $f_{Z|B_i}(z)$ is the same for every i. Thus, we can write

$$f_Z(z) = \sum_i P[B_i]\, f_{Z|B_i}(z) = f_{Z|B_0}(z) \sum_i P[B_i] = f_{Z|B_0}(z) \tag{5}$$

Thus, Z is a uniform $(-\Delta/2, \Delta/2)$ random variable. From the definition of a uniform (a, b) random variable, Z has mean and variance

$$E[Z] = 0, \qquad \text{Var}[Z] = \frac{(\Delta/2 - (-\Delta/2))^2}{12} = \frac{\Delta^2}{12}. \tag{6}$$

Problem 3.8.9 Solution

For this problem, almost any non-uniform random variable X will yield a non-uniform random variable Z. For example, suppose X has the "triangular" PDF

$$f_X(x) = \begin{cases} 8x/r^2 & 0 \le x \le r/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In this case, the event B_i that $Y = i\Delta + \Delta/2$ occurs if and only if $i\Delta \le X < (i+1)\Delta$. Thus

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} \frac{8x}{r^2}\, dx = \frac{8\Delta(i\Delta + \Delta/2)}{r^2} \tag{2}$$

It follows that the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} \frac{f_X(x)}{P[B_i]} & x \in B_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{x}{\Delta(i\Delta + \Delta/2)} & i\Delta \le x < (i+1)\Delta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Given event B_i, $Y = i\Delta + \Delta/2$, so that $Z = X - Y = X - i\Delta - \Delta/2$. This implies

$$f_{Z|B_i}(z) = f_{X|B_i}(z + i\Delta + \Delta/2) = \begin{cases} \frac{z + i\Delta + \Delta/2}{\Delta(i\Delta + \Delta/2)} & -\Delta/2 \le z < \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We observe that the PDF of Z depends on which event B_i occurs. Moreover, $f_{Z|B_i}(z)$ is non-uniform for all B_i.

Problem 3.9.7 Solution

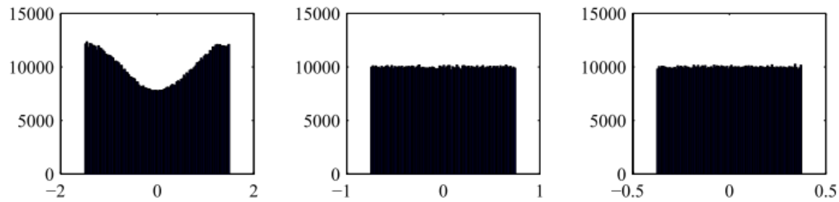
First we need to build a uniform $(-r/2, r/2)$ b-bit quantizer. The function `uquantize` does this.

```
function y=uquantize(r,b,x)
%uniform (-r/2,r/2) b bit quantizer
n=2^b;
delta=r/n;
x=min(x,(r-delta/2)/2);
x=max(x,-(r-delta/2)/2);
y=(delta/2)+delta*floor(x/delta);
```

Note that if $|x| > r/2$, then x is truncated so that the quantizer output has maximum amplitude. Next, we generate Gaussian samples, quantize them and record the errors:

```
function stdev=quantizegauss(r,b,m)
x=gaussrv(0,1,m);
x=x((x<=r/2)&(x>=-r/2));
y=uquantize(r,b,x);
z=x-y;
hist(z,100);
stdev=sqrt(sum(z.^2)/length(z));
```

For a Gaussian random variable X, $P[|X| > r/2] > 0$ for any value of r. When we generate enough Gaussian samples, we will always see some quantization errors due to the finite $(-r/2, r/2)$ range. To focus our attention on the effect of b bit quantization, `quantizegauss.m` eliminates Gaussian samples outside the range $(-r/2, r/2)$. Here are outputs of `quantizegauss` for $b = 1, 2, 3$ bits.



$b = 1$ $b = 2$ $b = 3$

It is obvious that for $b = 1$ bit quantization, the error is decidely not uniform. However, it appears that the error is uniform for $b = 2$ and $b = 3$. You can verify that uniform errors is a reasonable model for larger values of b.