Probability and Statistics HW2

1. (a) Prove Theorem 2.15(b) on the textbook.
   (b) Prove Theorem 2.15(c) on the textbook.
   (c) Prove Theorem 2.15(d) on the textbook.
   (d) Prove Theorem 2.15(e) on the textbook.

2. A sequence of characters is transmitted over a channel that introduces errors with probability $p = 0.01$.
   (a) What is the pmf of N, the number of error-free characters between erroneous characters?
   (b) What is $E[N]$?
   (c) Suppose we want to be 99% sure that at least 1000 characters are received correctly before a bad one occurs. What is the appropriate value of p?

3. Let N be a geometric random variable with $S_N = \{1, 2, \ldots\}$.
   (a) Find $P[N = k | N \leq m]$.
   (b) Find the probability that N is even.

4. An audio player uses a low-quality hard drive. The initial cost of building the player is \$50. The hard drive fails after each month of use with probability 1/12. The cost to repair the hard drive is \$20. If a 1-year warranty is offered, how much should the manufacturer charge so that the probability of losing money on a player is 1% or less? What is the average cost per player?

5. A data center has 10,000 disk drives. Suppose that a disk drive fails in a given day with probability 10^{-3}.
   (a) Find the probability that there are no failures in a given day.
   (b) Find the probability that there are fewer than 10 failures in two days.
   (c) Find the number of spare disk drives that should be available so that all failures in a day can be replaced with probability 99%.

6. A voltage X is uniformly distributed in the set $\{-3, \ldots, 3, 4\}$.
   (a) Find the mean and variance of X.
   (b) Find the mean and variance of $Y = -2X^2 + 3$.
   (c) Find the mean and variance of $W = \cos(\pi X/8)$.
   (d) Find the mean and variance of $Z = \cos^2(\pi X/8)$.

7. The fraction of defective items in a production line is p. Each item is tested and defective items are identified correctly with probability a.
   (a) Assume nondefective items always pass the test. What is the probability that k items are tested until a defective item is identified?
   (b) Suppose that the identified defective items are removed. What proportion of the remaining items is defective?
   (c) Now suppose that nondefective items are identified as defective with probability b. Repeat part (b).

8. The number X of photons counted by a receiver in an optical communication system is a Poisson random variable with rate λ_1 when a signal is present and a Poisson random variable with rate $\lambda_0 < \lambda_1$ when a signal is absent. Suppose that a signal is present with probability p.
   (a) Find $P[\text{signal present} | X = k]$ and $P[\text{signal absent} | X = k]$.

(b) The receiver uses the following decision rule:
If $P[\text{signal present}|X = k] > P[\text{signal absent}|X = k]$, decide signal present; otherwise, decide signal absent.
Show that this decision rule leads to the following threshold rule:
If $X > T$, decide signal present; otherwise, decide signal absent.

(c) What is the probability of error for the above decision rule?

9. A binary information source (e.g., a document scanner) generates very long strings of 0's followed by occasional 1's. Suppose that symbols are independent and that $p = P[\text{symbol}= 0]$ is very close to one. Consider the following scheme for encoding the run X of 0's between consecutive 1's:

(a) If $X = n$, express n as a multiple of an integer $M = 2^m$ and a remainder r, that is, find k and r such that $n = kM + r$, where $0 \leq r < M - 1$).

(b) The binary codeword for n then consists of a prefix consisting of k 0's followed by a 1, and a suffix consisting of the m-bit representation of the remainder r. The decoder can deduce the value of n from this binary string.

i. Find the probability that the prefix has k zeros, assuming that $p^M = 1/2$.

ii. Find the average codeword length when $p^M = 1/2$.

iii. Find the compression ratio, which is defined as the ratio of the average run length to the average codeword length when $p^M = 1/2$.