Probability and Statistics

Hw1 Solution

Problem 1.2.3 Solution

The sample space is

$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\}. \tag{1}$$

The event H is the set

$$H = \{A\heartsuit, \ldots, K\heartsuit\}. \tag{2}$$

Problem 1.3.4 Solution

Let s_i denote the outcome that the down face has i dots. The sample space is $S = \{s_1, \ldots, s_6\}$. The probability of each sample outcome is $P[s_i] = 1/6$. From Theorem 1.1, the probability of the event E that the roll is even is

$$P[E] = P[s_2] + P[s_4] + P[s_6] = 3/6. \tag{1}$$

Problem 1.4.4 Solution

Each statement is a consequence of part 4 of Theorem 1.4.

(a) Since $A \subset A \cup B$, $P[A] \le P[A \cup B]$.

(b) Since $B \subset A \cup B$, $P[B] \le P[A \cup B]$.

(c) Since $A \cap B \subset A$, $P[A \cap B] \le P[A]$.

(d) Since $A \cap B \subset B$, $P[A \cap B] \le P[B]$.

Problem 1.4.9 Solution

Each claim in Theorem 1.7 requires a proof from which we can check which axioms are used. However, the problem is somewhat hard because there may still be a simpler proof that uses fewer axioms. Still, the proof of each part will need Theorem 1.4 which we now prove.

For the mutually exclusive events $B_1, \ldots, B_m$, let $A_i = B_i$ for $i = 1, \ldots, m$ and let $A_i = \phi$ for $i > m$. In that case, by Axiom 3,

$$\begin{aligned} P[B_1 \cup B_2 \cup \cdots \cup B_m] &= P[A_1 \cup A_2 \cup \cdots] && (1)\\ &= \sum_{i=1}^{m-1} P[A_i] + \sum_{i=m}^{\infty} P[A_i] && (2)\\ &= \sum_{i=1}^{m-1} P[B_i] + \sum_{i=m}^{\infty} P[A_i]. && (3) \end{aligned}$$

Now, we use Axiom 3 again on $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^{\infty} P[A_i] = P[A_m \cup A_{m+1} \cup \cdots] = P[B_m]. \tag{4}$$

Thus, we have used just Axiom 3 to prove Theorem 1.4:

$$P[B_1 \cup B_2 \cup \cdots \cup B_m] = \sum_{i=1}^{m} P[B_i]. \tag{5}$$

(a) To show $P[\phi] = 0$, let $B_1 = S$ and let $B_2 = \phi$. Thus by Theorem 1.4,

$$P[S] = P[B_1 \cup B_2] = P[B_1] + P[B_2] = P[S] + P[\phi]. \tag{6}$$

Thus, $P[\phi] = 0$. Note that this proof uses only Theorem 1.4 which uses only Axiom 3.

(b) Using Theorem 1.4 with $B_1 = A$ and $B_2 = A^c$, we have

$$P[S] = P[A \cup A^c] = P[A] + P[A^c]. \tag{7}$$

Since, Axiom 2 says $P[S] = 1$, $P[A^c] = 1 - P[A]$. This proof uses Axioms 2 and 3.

(c) By Theorem 1.2, we can write both A and B as unions of disjoint events:

$$A = (AB) \cup (AB^c) \qquad B = (AB) \cup (A^c B). \tag{8}$$

Now we apply Theorem 1.4 to write

$$P[A] = P[AB] + P[AB^c], \qquad P[B] = P[AB] + P[A^c B]. \tag{9}$$

We can rewrite these facts as

$$P[AB^c] = P[A] - P[AB], \qquad P[A^c B] = P[B] - P[AB]. \tag{10}$$

Note that so far we have used only Axiom 3. Finally, we observe that $A \cup B$ can be written as the union of mutually exclusive events

$$A \cup B = (AB) \cup (AB^c) \cup (A^c B). \tag{11}$$

Once again, using Theorem 1.4, we have

$$P[A \cup B] = P[AB] + P[AB^c] + P[A^c B] \tag{12}$$

Substituting the results of Equation (10) into Equation (12) yields

$$P[A \cup B] = P[AB] + P[A] - P[AB] + P[B] - P[AB], \tag{13}$$

which completes the proof. Note that this claim required only Axiom 3.

(d) Observe that since $A \subset B$, we can write B as the disjoint union $B = A \cup (A^c B)$. By Theorem 1.4 (which uses Axiom 3),

$$P[B] = P[A] + P[A^c B]. \tag{14}$$

By Axiom 1, $P[A^c B] \geq 0$, hich implies $P[A] \leq P[B]$. This proof uses Axioms 1 and 3.

Problem 1.6.3 Solution

(a) Since A and B are disjoint, $P[A \cap B] = 0$. Since $P[A \cap B] = 0$,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 3/8. \tag{1}$$

A Venn diagram should convince you that $A \subset B^c$ so that $A \cap B^c = A$. This implies

$$P[A \cap B^c] = P[A] = 1/4. \tag{2}$$

It also follows that $P[A \cup B^c] = P[B^c] = 1 - 1/8 = 7/8$.

(b) Events A and B are dependent since $P[AB] \neq P[A]P[B]$.

(c) Since C and D are independent,

$$P[C \cap D] = P[C]\,P[D] = 15/64. \tag{3}$$

The next few items are a little trickier. From Venn diagrams, we see

$$P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64. \tag{4}$$

It follows that

$$P[C \cup D^c] = P[C] + P[D^c] - P[C \cap D^c] \tag{5}$$
$$= 5/8 + (1 - 3/8) - 25/64 = 55/64. \tag{6}$$

Using DeMorgan's law, we have

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64. \tag{7}$$

(d) Since $P[C^c D^c] = P[C^c]P[D^c]$, C^c and D^c are independent.

Problem 1.6.5 Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events

$$A_1 = \{1, 2\} \qquad A_2 = \{2, 3\} \qquad A_3 = \{3, 1\}. \tag{1}$$

Each event A_i has probability 1/2. Moreover, each pair of events is independent since

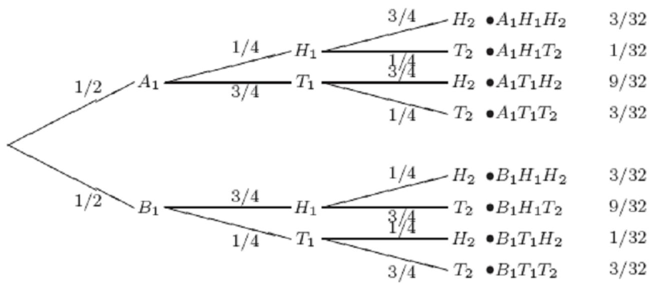
$$P[A_1A_2] = P[A_2A_3] = P[A_3A_1] = 1/4. \tag{2}$$

However, the three events A_1, A_2, A_3 are not independent since

$$P[A_1A_2A_3] = 0 \neq P[A_1] P[A_2] P[A_3]. \tag{3}$$

Problem 1.7.7 Solution

The tree for this experiment is



The event H_1H_2 that heads occurs on both flips has probability

$$P[H_1H_2] = P[A_1H_1H_2] + P[B_1H_1H_2] = 6/32. \tag{1}$$

The probability of H_1 is

$$P[H_1] = P[A_1H_1H_2] + P[A_1H_1T_2] + P[B_1H_1H_2] + P[B_1H_1T_2] = 1/2. \tag{2}$$

Similarly,

$$P[H_2] = P[A_1H_1H_2] + P[A_1T_1H_2] + P[B_1H_1H_2] + P[B_1T_1H_2] = 1/2. \tag{3}$$

Thus $P[H_1H_2] \neq P[H_1]P[H_2]$, implying H_1 and H_2 are not independent. This result should not be surprising since if the first flip is heads, it is likely that coin B was picked first. In this case, the second flip is less likely to be heads since it becomes more likely that the second coin flipped was coin A.

Problem 1.8.6 Solution

(a) We can find the number of valid starting lineups by noticing that the swingman presents three situations: (1) the swingman plays guard, (2) the swingman plays forward, and (3) the swingman doesn't play. The first situation is when the swingman can be chosen to play the guard position, and the second where the swingman can only be chosen to play the forward position. Let N_i denote the number of lineups corresponding to case i. Then we can write the total number of lineups as $N_1 + N_2 + N_3$. In the first situation, we have to choose 1 out of 3 centers, 2 out of 4 forwards, and 1 out of 4 guards so that

$$N_1 = \binom{3}{1}\binom{4}{2}\binom{4}{1} = 72. \tag{1}$$

In the second case, we need to choose 1 out of 3 centers, 1 out of 4 forwards and 2 out of 4 guards, yielding

$$N_2 = \binom{3}{1}\binom{4}{1}\binom{4}{2} = 72. \tag{2}$$

Finally, with the swingman on the bench, we choose 1 out of 3 centers, 2 out of 4 forward, and 2 out of four guards. This implies

$$N_3 = \binom{3}{1}\binom{4}{2}\binom{4}{2} = 108, \tag{3}$$

and the total number of lineups is $N_1 + N_2 + N_3 = 252$.

Problem 1.9.5 Solution

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3 \qquad P[G_2] = 2/3. \tag{1}$$

In addition, the problem statement tells us that

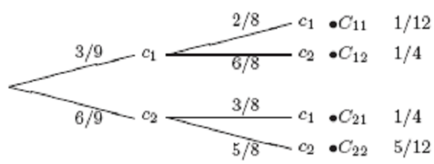
$$P[K|G_1] = 1/2 \qquad P[K|G_2] = 1/3. \tag{2}$$

Combining these facts using the Law of Total Probability yields

$$P[K] = P[K|G_1]\,P[G_1] + P[K|G_2]\,P[G_2] \tag{3}$$

$$= (1/2)(1/3) + (1/3)(2/3) = 7/18. \tag{4}$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and the second kicker from group j. The experiment to choose the kickers is described by the sample tree:



Since a kicker from group 1 makes a kick with probability 1/2 while a kicker from group 2 makes a kick with probability 1/3,

$$P[K_1K_2|C_{11}] = (1/2)^2 \qquad P[K_1K_2|C_{12}] = (1/2)(1/3) \tag{5}$$

$$P[K_1K_2|C_{21}] = (1/3)(1/2) \qquad P[K_1K_2|C_{22}] = (1/3)^2 \tag{6}$$

By the law of total probability,

$$P[K_1K_2] = P[K_1K_2|C_{11}]\,P[C_{11}] + P[K_1K_2|C_{12}]\,P[C_{12}] \tag{7}$$

$$+ P[K_1K_2|C_{21}]\,P[C_{21}] + P[K_1K_2|C_{22}]\,P[C_{22}] \tag{8}$$

$$= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} = \frac{65}{432}\ . \tag{9}$$

It should be apparent that $P[K_1] = P[K]$ from part (a). Symmetry should also make it clear that $P[K_1] = P[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $P[K_2|C_{ij}]$ and using the law of total probability to calculate $P[K_2]$.

$$P[K_2|C_{11}] = 1/2, \qquad P[K_2|C_{12}] = 1/3, \tag{10}$$

$$P[K_2|C_{21}] = 1/2, \qquad P[K_2|C_{22}] = 1/3. \tag{11}$$

By the law of total probability,

$$\begin{aligned}P[K_2] &= P[K_2|C_{11}]P[C_{11}] + P[K_2|C_{12}]P[C_{12}] \\ &\quad + P[K_2|C_{21}]P[C_{21}] + P[K_2|C_{22}]P[C_{22}] \qquad (12)\\ &= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} = \frac{7}{18}. \qquad (13)\end{aligned}$$

We observe that K_1 and K_2 are not independent since

$$P[K_1K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = P[K_1]P[K_2]. \qquad (14)$$

Note that $15/96$ and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss.

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is $1/2$. If the kicker is from group 2, the success probability is $1/3$. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

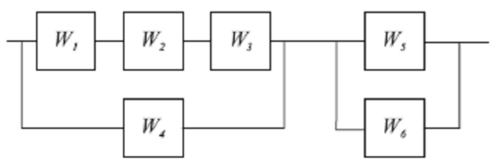
$$P[M|G_1] = \binom{10}{5}(1/2)^5(1/2)^5, \qquad P[M|G_2] = \binom{10}{5}(1/3)^5(2/3)^5. \qquad (15)$$

We use the Law of Total Probability to find

$$\begin{aligned}P[M] &= P[M|G_1]P[G_1] + P[M|G_2]P[G_2] \qquad (16)\\ &= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right). \qquad (17)\end{aligned}$$

Problem 1.10.4 Solution

From the statement of Problem 1.10.1, the configuration of device components is



By symmetry, note that the reliability of the system is the same whether we replace component 1, component 2, or component 3. Similarly, the reliability is the same whether we replace component 5 or component 6. Thus we consider the following cases:

I **Replace component 1** In this case

$$P[W_1W_2W_3] = (1-\frac{q}{2})(1-q)^2, \qquad P[W_4] = 1-q, \qquad P[W_5 \cup W_6] = 1-q^2. \qquad (1)$$

This implies

$$P[W_1W_2W_3 \cup W_4] = 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) = 1 - \frac{q^2}{2}(5 - 4q + q^2). \qquad (2)$$

In this case, the probability the system works is

$$P[W_I] = P[W_1W_2W_3 \cup W_4]P[W_5 \cup W_6] = \left[1 - \frac{q^2}{2}(5-4q+q^2)\right](1-q^2). \qquad (3)$$

II **Replace component 4** In this case,

$$P[W_1W_2W_3] = (1-q)^3, \qquad P[W_4] = 1 - \frac{q}{2}, \qquad P[W_5 \cup W_6] = 1 - q^2. \qquad (4)$$

This implies

$$P[W_1W_2W_3 \cup W_4] = 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) = 1 - \frac{q}{2} + \frac{q}{2}(1-q)^3. \qquad (5)$$

In this case, the probability the system works is

$$P[W_{II}] = P[W_1W_2W_3 \cup W_4]\, P[W_5 \cup W_6] = \left[1 - \frac{q}{2} + \frac{q}{2}(1-q)^3\right](1-q^2). \qquad (6)$$

III Replace component 5 In this case,

$$P[W_1W_2W_3] = (1-q)^3, \qquad P[W_4] = 1-q, \qquad P[W_5 \cup W_6] = 1 - \frac{q^2}{2}. \qquad (7)$$

This implies

$$P[W_1W_2W_3 \cup W_4] = 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) = (1-q)\left[1 + q(1-q)^2\right]. \qquad (8)$$

In this case, the probability the system works is

$$\begin{aligned} P[W_{III}] &= P[W_1W_2W_3 \cup W_4]\, P[W_5 \cup W_6] && (9)\\ &= (1-q)\left(1 - \frac{q^2}{2}\right)\left[1 + q(1-q)^2\right]. && (10) \end{aligned}$$

From these expressions, its hard to tell which substitution creates the most reliable circuit. First, we observe that $P[W_{II}] > P[W_I]$ if and only if

$$1 - \frac{q}{2} + \frac{q}{2}(1-q)^3 > 1 - \frac{q^2}{2}(5 - 4q + q^2). \qquad (11)$$

Some algebra will show that $P[W_{II}] > P[W_I]$ if and only if $q < 2$, which occurs for all nontrivial (i.e., nonzero) values of q. Similar algebra will show that $P[W_{II}] > P[W_{III}]$ for all values of $0 \le q \le 1$. Thus the best policy is to replace component 4.

Problem 1.11.4 Solution

To test n 6-component devices, (such that each component works with probability q) we use the following function:

```
function N=reliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
W=rand(n,6)>q;
D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
D=D&(W(:,5)|W(:,6));
N=sum(D);
```

The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical operators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Lastly, we count the number `N` of working devices. The following code snippet produces ten sample runs, where each sample run tests `n=100` devices for $q = 0.2$.

```
>> for n=1:10, w(n)=reliable6(100,0.2); end
>> w
w =
    82    87    87    92    91    85    85    83    90    89
>>
```

As we see, the number of working devices is typically around 85 out of 100. Solving Problem 1.10.1, will show that the probability the device works is actually 0.8663.