# Probability and Statistics

## Quiz Two Solution

1.

*Sol* :

$$\begin{cases} X = R\cos\Phi \\ Y = R\sin\Phi \end{cases} \Rightarrow \begin{cases} R = \sqrt{X^2 + Y^2} \\ \Phi = \tan^{-1}\dfrac{Y}{X} \end{cases}$$

$\because$ $X$ and $Y$ are two i.i.d. Gaussian random variables with zero mean and unit variance.

$\therefore$ The joint PDF $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \dfrac{1}{2\pi}\exp\{-\dfrac{x^2+y^2}{2}\}$.

By Jacobian transformation, we can obtain the joint PDF as following:

$$f_{R,\Phi}(r,\phi) = f_{X,Y}(x,y)\cdot\left|\frac{dxdy}{drd\phi}\right| = \frac{1}{2\pi}\exp\{-\frac{r^2}{2}\}\cdot r,\ \ r \ge 0.$$

(1) The PDF of $R$:

$$f_R(r) = \int_{-\pi}^{\pi} f_{R,\Phi}(r,\phi)d\phi = r\cdot\exp\{-\frac{r^2}{2}\}u(r).$$

(2) The PDF of $\Phi$:

$$f_\Phi(\phi) = \int_0^\infty f_{R,\Phi}(r,\phi)d\phi = \frac{1}{2\pi},\ \ \phi\in[-\pi,\pi).$$

$\because$ $f_{R,\Phi}(r,\phi) = \dfrac{1}{2\pi}\cdot r\cdot\exp\{-\dfrac{r^2}{2}\}u(r) = f_R(r)\cdot f_\Phi(\phi)$.

$\therefore$ $R$ and $\Phi$ are independent.

2.

*Sol* :

$\because$ $X$ and $Y$ are two i.i.d. exponential random variables.

$\therefore$ The joint PDF $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = \exp\{-(x+y)\}u(x)u(y)$.

Let $A = X - Y$, such that $Z = (X-Y)u(X-Y) = Au(A)$.

$$\begin{aligned}
\text{The PDF } f_A(a) &= \int_0^\infty f_{X,Y}(a+y,y)dy \\
&= \int_0^\infty \exp\{-(a+2y)\}u(a+y)u(y)dy \\
&= \exp\{-a\}\cdot\begin{cases} \int_0^\infty \exp\{-2y\}dy, & a \ge 0 \\ \int_{-a}^\infty \exp\{-2y\}dy, & a < 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}\exp\{-a\}, & a \ge 0 \\ \frac{1}{2}\exp\{a\}, & a < 0 \end{cases} = \frac{1}{2}\exp\{-|a|\}
\end{aligned}$$

$\because Z = Au(A) = \begin{cases} A, & A \geq 0 \\ 0, & A < 0 \end{cases}$

$\therefore$ The PDF $f_Z(z) = \frac{1}{2}\delta(z) + \frac{1}{2}\exp\{-z\}u(z)$

where $\delta(z) = \begin{cases} \infty, & z = 0 \\ 0, & z \neq 0 \end{cases}$ with $\int_{-\infty}^{\infty} \delta(z)dz = 1.$

$\Rightarrow E[z] = \int_{-\infty}^{\infty} z \cdot f_Z(z)dz = \frac{1}{2}.$

3.

*Sol* :

Define $Z = \max[X_1, X_2, ..., X_n]$, and $W = \min[X_1, X_2, ..., X_n]$.

The joint CDF $F_{Z,W}(z,w) = P[Z \leq z, W \leq w]$

$= P[Z \leq z] - P[Z \leq z, W > w]$

(1) $\because X_1, X_2, ..., X_n$ are i.i.d. random variables with PDF $f_X(x)$ and CDF $F_X(x)$.

$\therefore P[Z \leq z] = [F_X(z)]^n.$

(2) $P[Z \leq z, W > w] = P[\max[X_1, X_2, ..., X_n] \leq z, \min[X_1, X_2, ..., X_n] > w]$

$= P[w < X_1 \leq z, w < X_2 \leq z, ..., w < X_n \leq z]$

$= P[w < X_1 \leq z] \cdot P[w < X_2 \leq z] \cdot ... \cdot P[w < X_n \leq z]$

$= [F_X(z) - F_X(w)]^n.$

(3) $F_{Z,W}(z,w) = P[Z \leq z] - P[Z \leq z, W > w]$

$= [F_X(z)]^n - [F_X(z) - F_X(w)]^n.$

The joint PDF $f_{Z,W}(z,w) = \frac{\partial^2 F_{Z,W}(z,w)}{\partial z \partial w}$

$= n(n-1)[F_X(z) - F_X(w)]^{n-2} f_X(z) f_X(w).$