CHAP 6

便箋標題 2008/5/15

$f * g(x)$ is called the convolution of $f(x)$ and $g(x)$ iff

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

$$= \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

By Taylor series expansion,

$$\Phi_X(\omega) = E[e^{sX}] = E\left[\sum_{n=0}^{\infty} (sX)^n / n!\right]$$

$$= \sum_{n=0}^{\infty} \frac{s^n}{n!} \underbrace{E[X^n]}_{\overset{\|}{\left.\frac{d^n\Phi_X(s)}{ds^n}\right|_{s=0}}}$$

Defn: $\Phi_X(\omega) \triangleq E[e^{j\omega X}]$ is called the characteristic function (CF) of X.

Notes: ① The MGF $\Phi_X(s)$ may exist only for certain region of s, i.e., the region of convergence.

② The CF $\Phi_X(\omega)$ exists for all ω values since

$$|\Phi_X(\omega)| = \left|\int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx\right|$$

$$\leq \int_{-\infty}^{\infty} \underbrace{|f_X(x) e^{j\omega x}|}_{= |f_X(x)| = f_X(x)} dx$$

$$= 1$$

③ $\Phi_X(-\omega) = \int_{-\infty}^{\infty} f_X(x) e^{-j\omega x} dx$ is the Fourier transform of $f_X(x)$.

Because $f_X(x)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |f_X(x)| dx$ exists, its Fourier transform $\Phi_X(-\omega)$ exists for all ω values.

Proof of Central Limit Theorem (CLT):

CLT: When X_n's are iid with identical mean μ_X and identical variance σ_X^2, we define the new random variables

$$U_n \triangleq \frac{X_n - \mu_X}{\sigma_{X_n}}, \ \forall n, \quad \text{and}$$

$$Y_n \triangleq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i$$

and obtain according to CLT that

$$\lim_{n\to\infty} F_{Y_n}(y) = \Phi(y).$$

Proof: Let us check the moment generating function of Y_n:

$$\phi_{Y_n}(s) \triangleq E[\exp(sY_n)]$$

$$= E[\exp(s\frac{1}{\sqrt{n}}\sum_{i=1}^{n} U_i)]$$

U_i are iid with mean $= 0$, variance $= 1$ and MGF $\phi_M(s)$

$$= \prod_{i=1}^{n} E[\exp(\frac{s}{\sqrt{n}} U_i)]$$

$$= [\underbrace{\phi_M(\frac{s}{\sqrt{n}})}]^n \quad \cdots \circledast \text{ on}$$

Expanding $\phi_M(\frac{s}{\sqrt{n}})$ in a Taylor series, we have

$$\boxed{\phi_M(\frac{s}{\sqrt{n}})} \triangleq E[\exp(\frac{s}{\sqrt{n}} U)]$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \overset{\curvearrowright}{=} \sum_{i=0}^{\infty} \frac{1}{i!} E\left[\left(\frac{s}{\sqrt{n}} u\right)^i\right]$$

$$= 1 + \frac{s}{\sqrt{n}} \underbrace{E[u]}_{=0} + \frac{s^2}{2n} \underbrace{E[u^2]}_{=1} + \underbrace{\frac{s^3}{6n^{3/2}} E[u^3] + \cdots}$$

$$= 1 + \frac{s^2}{2n} + \boxed{\frac{1}{n} R(s,n)}$$

Where $R(s,n) \equiv \frac{s^3}{6n^{1/2}} E[u^3] + \frac{s^4}{24n} E[u^4] + \cdots$

and $\lim_{n \to \infty} R(s,n) = 0$.

Next, substituting the above series into (*) gives

$$\phi_{Y_n}(s) = \left[1 + \frac{s^2}{2n} + \frac{1}{n} R(s,n)\right]^n$$

Taking the natural logarithm further gives

$$\ln \phi_{Y_n}(s) = n \underbrace{\ln\left[1 + \left(\frac{s^2}{2n} + \frac{1}{n} R(s,n)\right)\right]}$$

For small $|x|$, i.e., $|x| \ll 1$,

$$\ln(1+x) = \underbrace{x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} x^i$$

which expands $\ln \phi_{Y_n}(s)$ into

$$\ln \phi_{Y_n}(s) = \cancel{n}\left[\left(\frac{s^2}{2n} + \frac{1}{n} R(s,n)\right) - \underbrace{\frac{1}{2}\left(\frac{s^2}{2n} + \frac{1}{n} R(s,n)\right)^2}_{\frac{1}{n^2}} + \cdots\right]$$

$$= \frac{s^2}{2} + \underbrace{R(s,n)}_{\xrightarrow{\text{as } n\to\infty} 0} + \underbrace{\frac{1}{n}\left[\boxed{- - -}\right]}_{\xrightarrow{\text{as } n\to\infty} 0}$$

Thus, $\lim_{n\to\infty} \Phi_{Y_n}(s) = e^{\frac{s^2}{2}}$ which is the moment generating function of the standard normal random variable; i.e., $\lim_{n\to\infty} F_{Y_n}(y) = \Phi(y)$. Q.E.D.

Note: There are several versions of CLT. For example, there exist versions for correlated sequences. For more exposure read the reference

P. Billingsley, "Probability and Measure," 2nd ed, Wiley, 1986 (Section 27).

When X is continuous, $\boxed{e^{-x}, x \geq 0}$

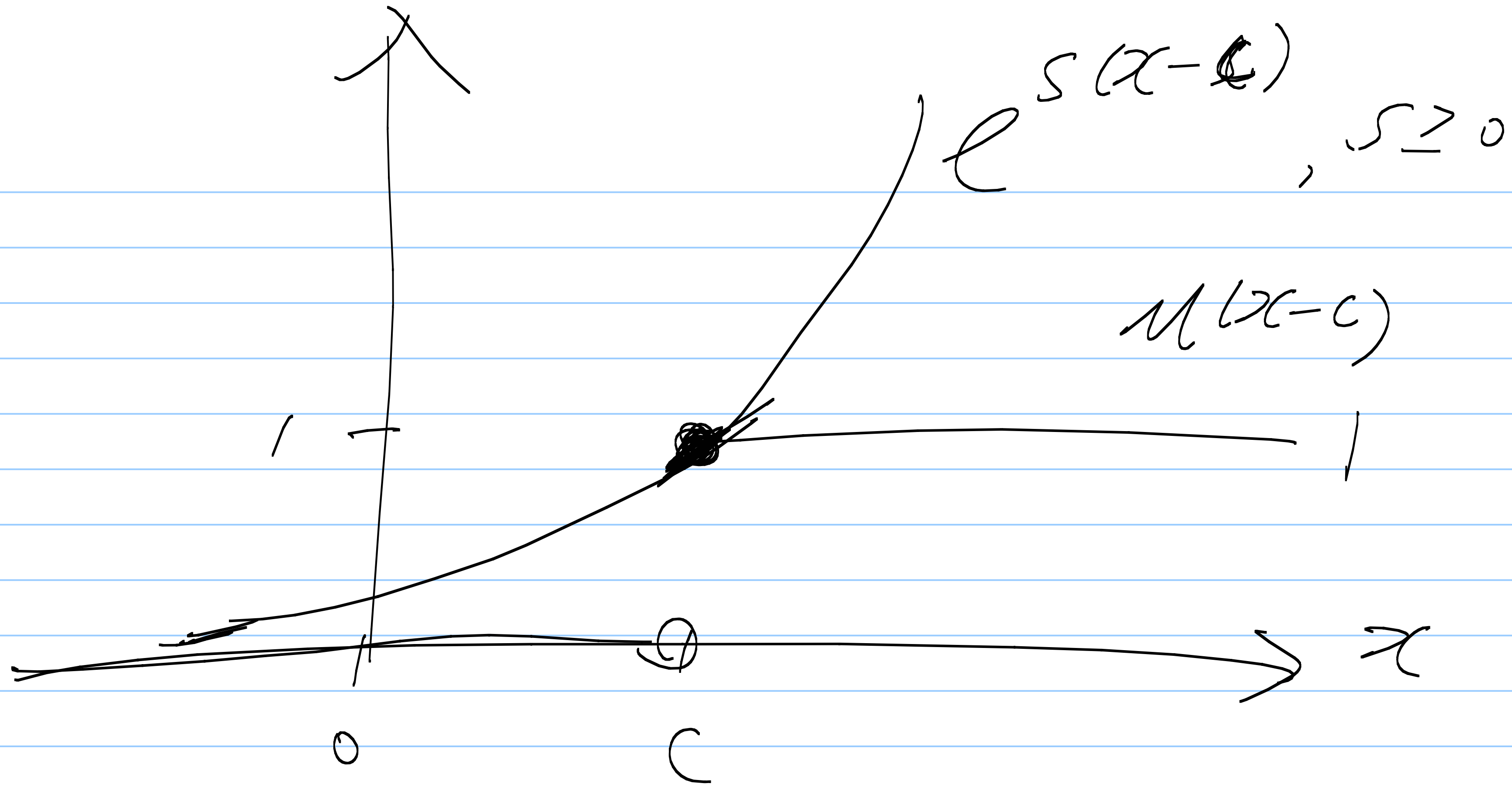
$$\phi_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx = \underbrace{\int_0^{\infty} e^{-(1-s)x}\, dx}_{\text{exists only } 1>s.}$$

$$\Rightarrow \quad \phi_X(-s) = \underset{\uparrow \text{Laplace Transform}}{\mathcal{L}}\{f_X(x)\}$$

Thus, $\phi_X(s)$ and $f_X(x)$ are a unique pair.

When X is discrete,

$$\phi_X(s) = \sum_{x \in S_X} e^{sx} \phi_X(s)$$



$$e^{s(x-c)} \geq u(x-c), \quad \forall s \geq 0$$

with "=" holding only when $x = c$.