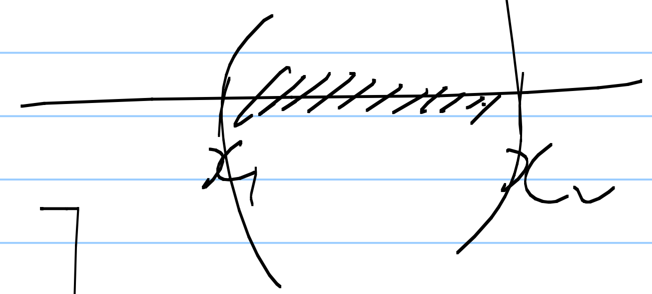
CHAP 3

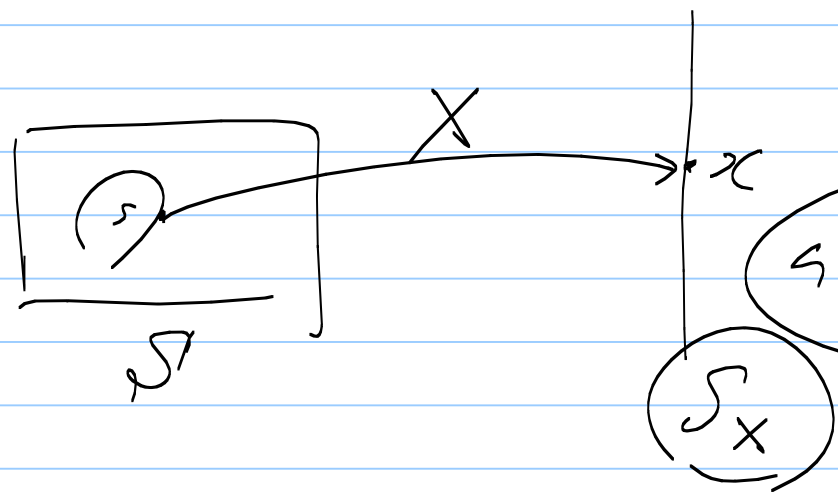
$$(x_1, x_2) \equiv \{x \mid x_1 < x < x_2\}, \quad x_1 < x_2$$

$[\quad]$ $\qquad \le \qquad \le$



$P[X = x] = 0$ if X is continuous

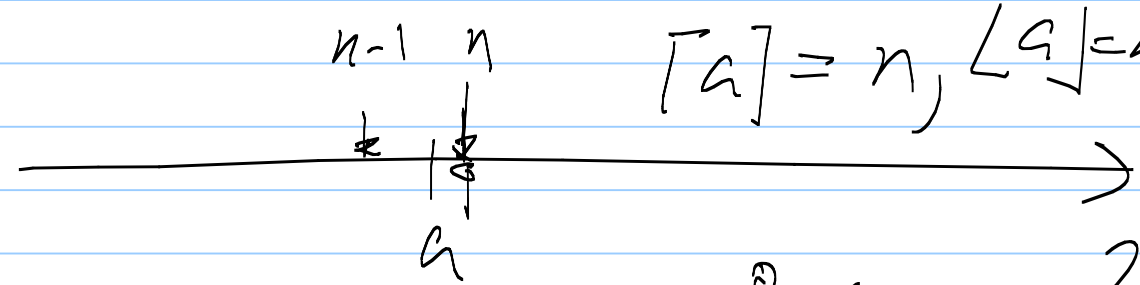
countable or finite
discrete



a set of continuous numbers

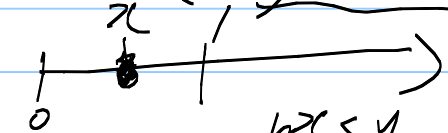
$\lceil a \rceil \equiv$ the smallest integer that is not smaller than a

$\lfloor a \rfloor \equiv$ the largest integer that is not larger than a

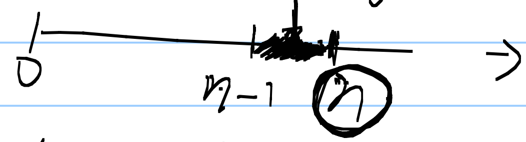


$\lceil a \rceil = n, \ \lfloor a \rfloor = n-1$

$\boxed{y = \lceil nx \rceil}$



$nx \le y, \ \lceil 0 \rceil$



$$p_Y(y) = \begin{cases} 1/n, & y = 1, 2, \dots, n \\ 0, & \text{elsewhere} \end{cases}$$

$Y = \lceil nX \rceil$

$\{X = x\} \subset \{Y = \lceil nx \rceil\}$

$P[X = x] \le \underbrace{P[Y = \lceil nx \rceil]}_{=}$

$\frac{1}{n}$

$\Rightarrow \mathbb{P}[X = x] \leq \frac{1}{n}$

PMF

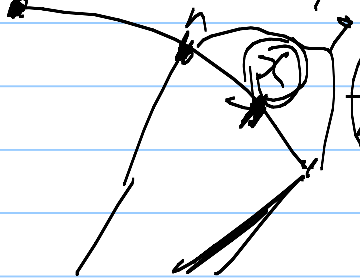
$\Rightarrow \lim_{n \to \infty} \underbrace{\mathbb{P}[X = x]} \leq 0$

$\underbrace{F_X(x)}_{\text{CDF}} = \underbrace{\mathbb{P}[X \leq x]}$

$\Rightarrow \mathbb{P}[X = x] \leq 0 \Rightarrow \mathbb{P}[X = x] = 0$.

$\frac{y-1}{n}$ $Y = y$ $y \in \{1, 2, \dots, n\}$

$\lceil nx \rceil = y$

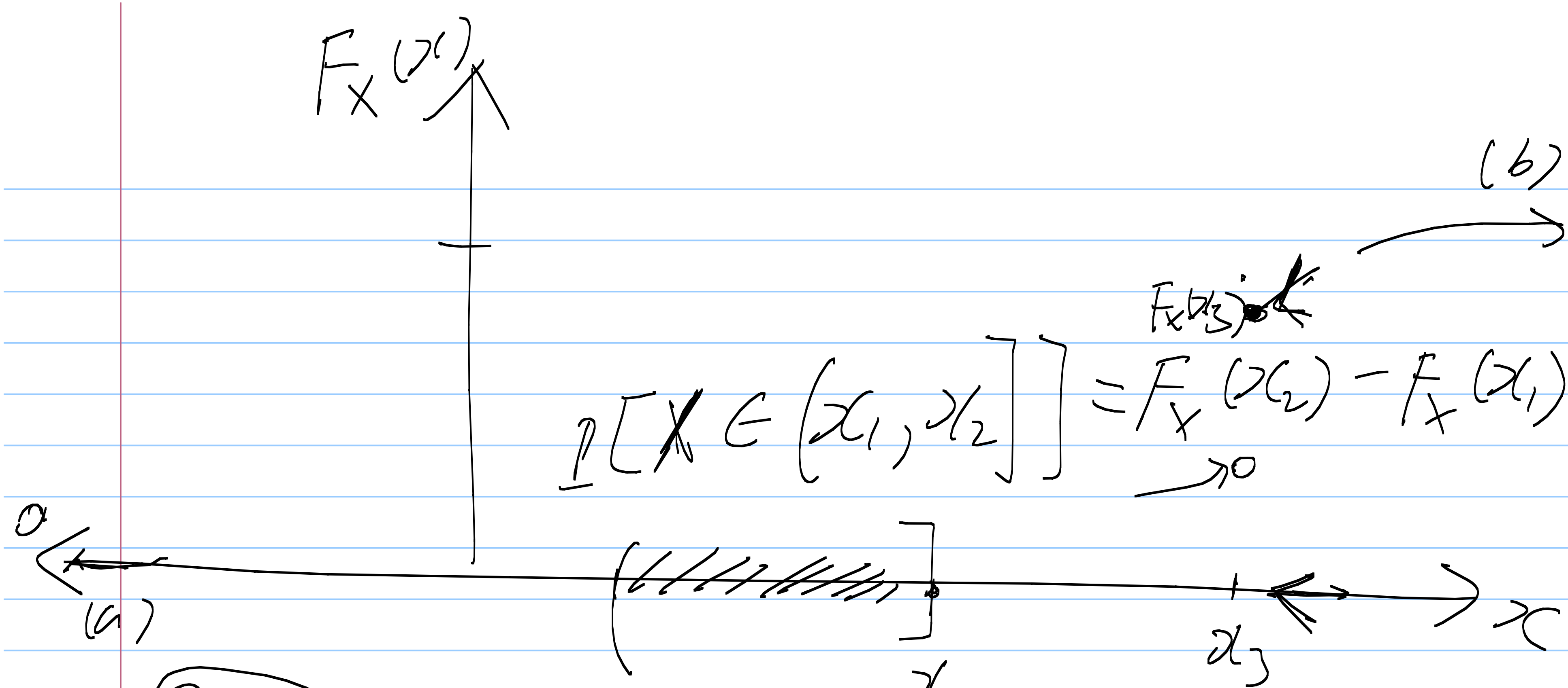


$\frac{y}{n}$

y

if $\underbrace{X = x}_{A}$, then $\underbrace{Y = \lceil nx \rceil}_{B}$.

$\{X = x\} \subset \{Y = \lceil nx \rceil\}$



$P[X \in (x_1, x_2]] = F_X(x_2) - F_X(x_1) \to 0$

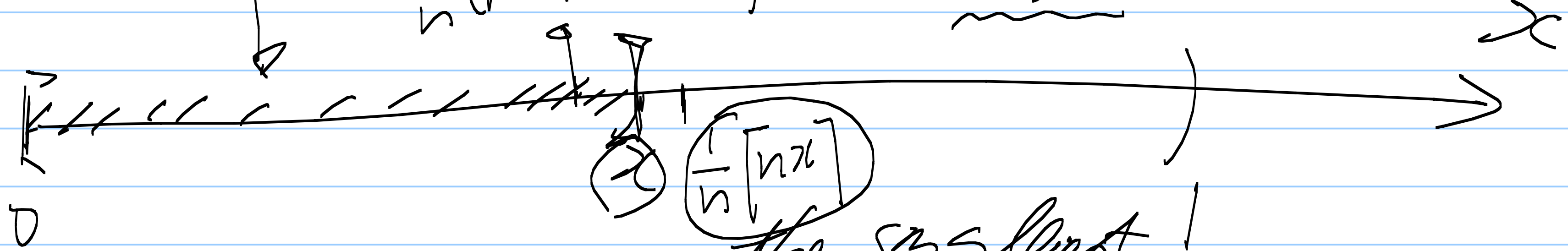
$F_X(x) \triangleq P[X \leq x]$ is continuous from the right.

Redefine Definition 3.2 : $S_X = [1, 2]$
$(1, 2)$

X is a continuous random variable if the CDF $F_X(x)$ is a continuous function, (i.e., $F_X(x)$ is continuous from the right and from the left as well for all $x \in S_X$), and <u>if</u> the first derivative of $F_X(x)$ exists for all x being in the interior of S_X.

$\{X \le x\} = \{nX \le nx\} \subset \{Y \le \lceil nx \rceil\}$

$\{Y \le \lceil nx \rceil\}$



the smallest!

$y = \lceil nx \rceil$ denotes y is ~~an~~ integer which is not smaller than nx.

$\{Y \le \lceil nx \rceil\}$

$\{Y \le \lceil nx \rceil - 1\} \subset \{nX \le nx\} \subset \{Y \le \lceil nx \rceil\}$

$\Delta > 0$

$$P_1 = \underbrace{P\left[X \in (x_1, x_1+\Delta]\right]}_{\displaystyle = F_X(x_1+\Delta) - F_X(x_1)} \gtrless \underbrace{P\left[X \in (x_2, x_2+\Delta]\right]}_{\displaystyle = F_X(x_2+\Delta) - F_X(x_2)} = P_2$$

$$= \frac{F_X(x_1+\Delta) - F_X(x_1)}{\Delta}\,\Delta \qquad \frac{F_X(x_2+\Delta) - F_X(x_2)}{\Delta}\,\Delta$$

$\Delta \to 0$

dx

$$\underbrace{\left(\lim_{\Delta\to 0}\frac{F_X(x_1+\Delta) - F_X(x_1)}{\Delta}\right)}_{= f_X(x_1)} dx = f_X(x_1)\,dx \qquad \underbrace{\left(\lim_{\Delta\to 0}\frac{F_X(x_2+\Delta) - F_X(x_2)}{\Delta}\right)}_{= f_X(x_2)} dx = f_X(x_2)\,dx$$

If $f_X(x_2) > f_X(x_1)$, then we say that the continuous random variable X has a higher likelihood to locate near x_2 than x_1. Also, $P[X \in (x_2, x_2 + dx]] = f_X(x_2)\,dx$

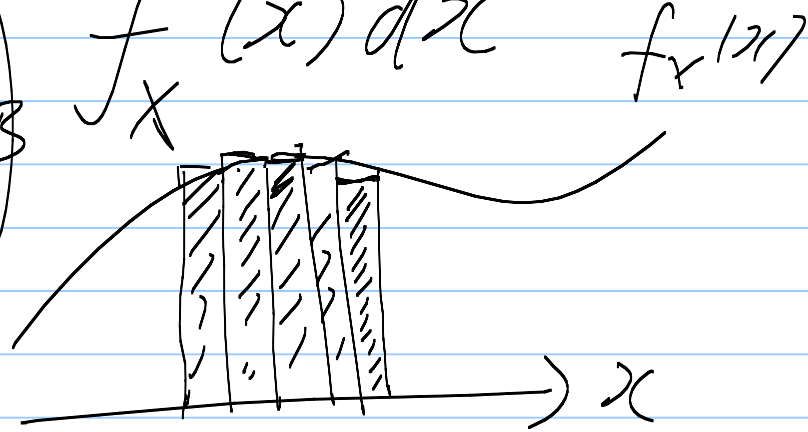
The slope at any point x indicates the probability that X is near x.

For discrete X,

$$P[X \in B] = \sum_{x \in B} p_X(x)$$

For continuous X,

$$P[X \in B] = \int_{x \in B} f_X(x)\,dx$$



For discrete Y,

$$E[Y] = \sum_{y \in S_Y} y \, P_Y(y)$$

$\{Y = k\Delta\} = \{k\Delta \le X < (k+1)\Delta\}$

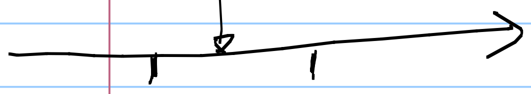
If we have a continuous rv X and

$$Y = \Delta \left\lfloor \frac{X}{\Delta} \right\rfloor \text{ for some } \Delta > 0, \text{ then}$$

$x = \frac{x}{\Delta}\Delta$

$$E[Y] = \sum_{k=-\infty}^{\infty} \underbrace{k\Delta}_{y} \, \underbrace{P_Y(k\Delta)}_{= P[Y = k\Delta] = P[\underbrace{k\Delta \le X < k\Delta + \Delta}_{k \le \frac{X}{\Delta} < k+1}]}$$

$k = \left\lfloor \frac{X}{\Delta} \right\rfloor$



$\underbrace{\left\lfloor \frac{x}{\Delta} \right\rfloor \Delta}_{= x \text{ as } \Delta \to 0}$ $\quad \left(\left\lfloor \frac{x}{\Delta} \right\rfloor + 1\right)\Delta$

$$= \sum_{k=-\infty}^{\infty} k\Delta \, \underbrace{P[k\Delta \le X < k\Delta + \Delta]}$$

$$\simeq E[X]$$

$$\Rightarrow E[X] \cong \sum_{k=-\infty}^{\infty} \underbrace{k\Delta}_{x} f_X(\underbrace{k\Delta}_{x})\Delta \qquad \underbrace{f_X(x)\,dx}_{\cancel{2}}$$

$$\overset{\Delta \to 0}{=} \boxed{\int_{-\infty}^{\infty} x f_X(x)\,dx} = \int_{-\infty}^{\infty} x\,dF_X(x)$$

If X is a continuous random variable

$$\underbrace{E[(X-\mu_X)^2]} = Var[X] \overset{\text{Thm 3.6}}{=} \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

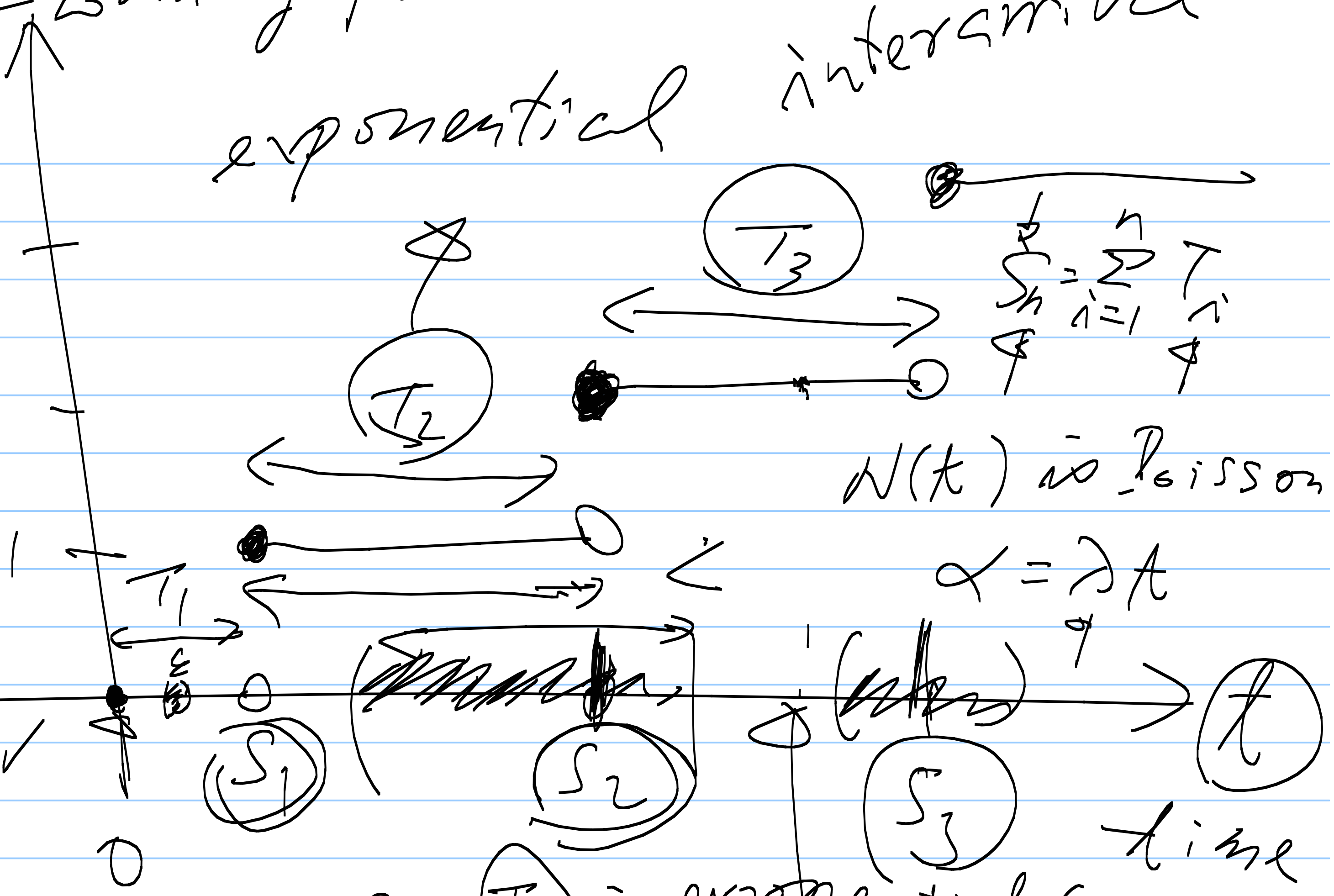
$$= \underbrace{\int_{-\infty}^{\infty} x^2 f_X(x)\,dx}_{= E[X^2]} - \Big[\underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{= E[X]}\Big]^2$$

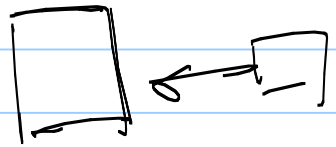
$\lambda t = \alpha$ $N(t)$ = counting process

exponential interarrival

1. increase by unit jump
2. stationary increment ✓
3. independent increment ✓
4. ✓

toll station





$S_n = \sum_{i=1}^{n} T_i$

$N(t)$ is Poisson

$\alpha = \lambda t$

$S_2 - S_1 = T_2$ is exponential(λ)

Erlang $S_3 - S_2 = T_3$

t S_n Erlang$^{(n,\lambda)}$

If X is Erlang(n, λ),

X_i is Exponential(λ), and

$X_1, X_2, \ldots, X_n$ are independent,

then X and $\sum_{i=1}^{n} X_i$ have the same PDF.

$$\underbrace{X}_{\text{Erlang}(n,\lambda)} = \sum_{i=1}^{n} \underbrace{X_i}_{\text{exponential}(\lambda)}$$

$E[X] = \sum_{i=1}^{n} E[X_i] = \frac{n}{\lambda}$

$Var[X] = \sum_{i=1}^{n} Var[X_i] = \frac{n}{\lambda^2}$

X is Gaussian, or normal, if

$$f_X(x) = \boxed{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

① $f_X(x)$ is symmetric with respect to $x = \mu$.

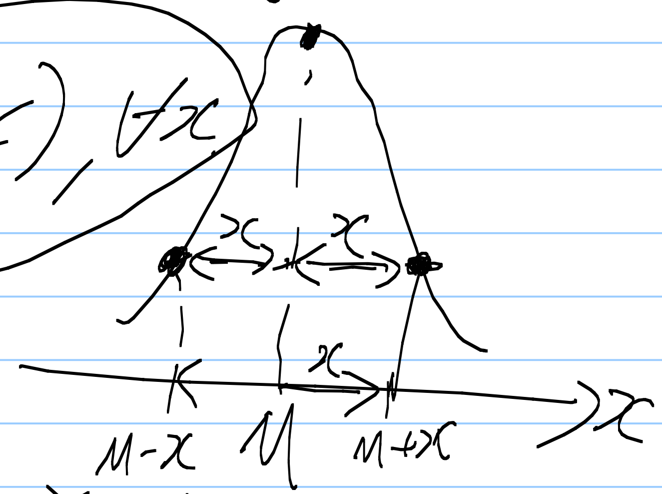
② $(x-\mu)^2$ is quadratic in $x-\mu$.

X is $N[\mu, \sigma^2]$,

$\equiv X$ is a $\boxed{\text{Gaussian}(\mu, \sigma)}$ random variable

① $f_X(x)$ is symmetric about $x = \mu$, i.e.

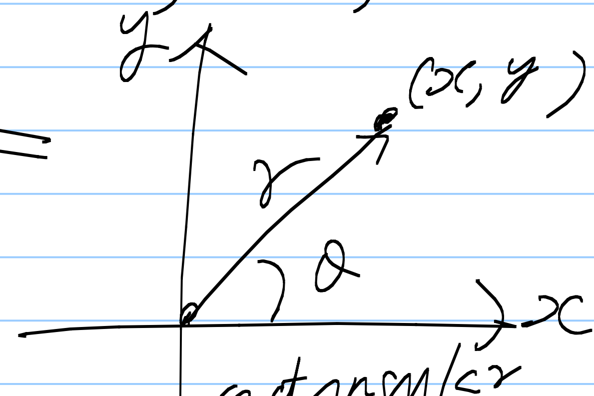
$$f_X(\mu + x) = f_X(\mu - x), \ \forall x$$



② $f_X(x)$ has a peak on $x = \mu$, i.e.,

$$\mu = \arg\left(\boxed{\max_x f_X(x)}\right)$$

$f_X(x)$ is unimodal.



$$\iint f(x,y)\, \underbrace{dx\,dy}_{r\,dr\,d\theta}$$

$$\iint f(r,\theta)\, r\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

Jacobian $dx\,dy = r\,dr\,d\theta$

rectangular

polar

$\Rightarrow Y = aX + b, \quad X \sim \underbrace{N[\mu, \sigma^2]}$

$$F_Y(y) = P[Y \leq y] = P[\underbrace{aX + b} \leq y] \qquad \text{if } a \neq 0$$

$$= P\left[X \leq \frac{y-b}{a}\right] \qquad a > 0 \qquad a < 0$$

$$= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} P[Y \leq y] = \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi a^2 \sigma^2}} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}}$$

Q.E.D.

If X is $\mathcal{N}[\mu, \sigma^2]$, then

① $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz$

$y = \frac{z-\mu}{\sigma}$ $= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}} dy$

$= \Phi\left(\frac{x-\mu}{\sigma}\right) \sim \mathcal{N}[0,1]$ Q.E.D.

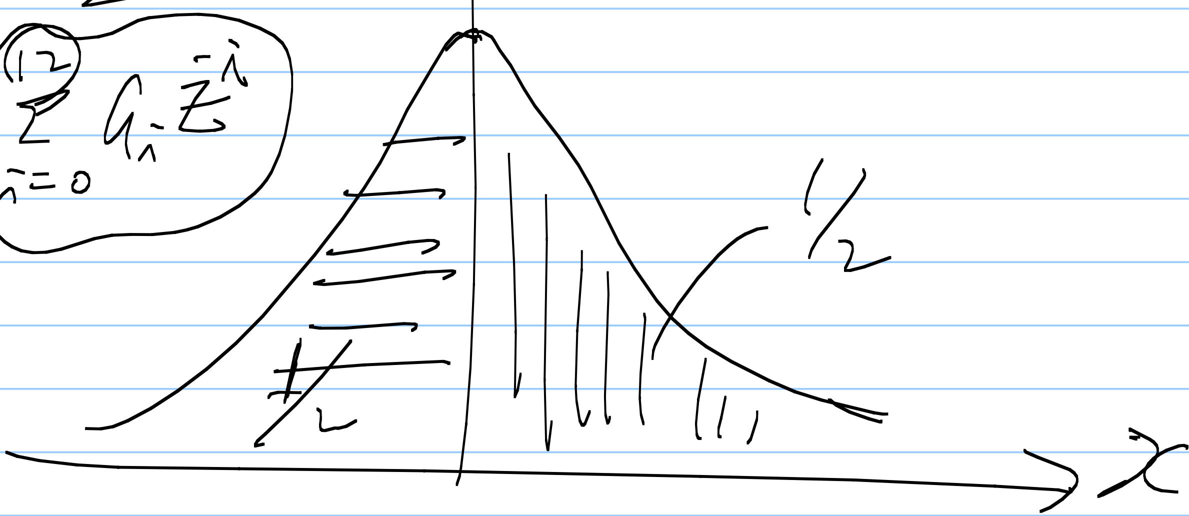
② $P[a < X \le b] = P\left[\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$

From Theorem 3.13, $\frac{X-\mu}{\sigma}$ is $N[0, 1]$, i.e., X is standard normal.

$f_X(x) = N[0,1]$

① $\Phi(0) = \frac{1}{2}$

$\Phi(z) = \sum_{i=0}^{12} a_i z^{-i}$



$\Phi(-z_0) = 1 - \Phi(z_0)$, $\forall z$

Pf: $\Phi(z) = \int_{-\infty}^{z} f_Z(x)\,dx$

$\Phi(-z) = \int_{-\infty}^{-z} f_Z(x)\,dx$

$y = -x$

$f_Z(x) = f_Z(-x)$

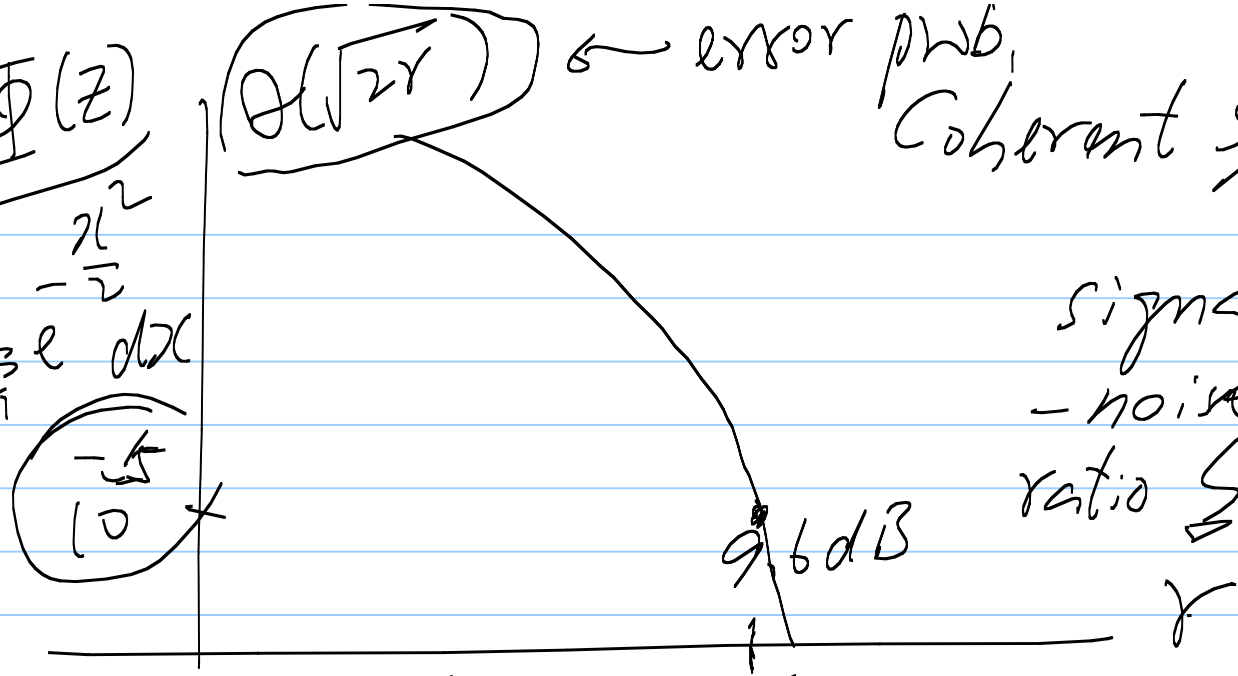
$= \int_{z}^{\infty} f_Z(y)\,dy$

$= 1 - \Phi(z)$, $\forall z$

Q.E.D.

Note: $\Phi(0) = 1/2$.

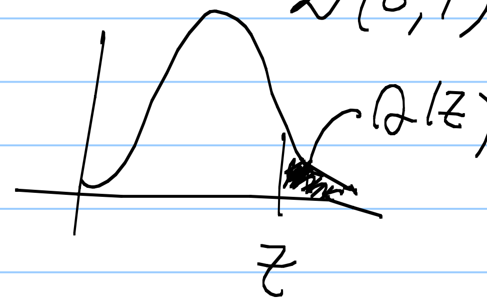
$$Q(z) \triangleq \underbrace{1 - \Phi(z)}_{} = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \qquad (z \ge 3)$$



← error prob.
Coherent System

signal-to-noise power ratio γ

$Q(z)$ is called the standard normal complementary CDF, Q function, or Gaussian tail integral.



$\delta(x) \equiv$ (Dirac) delta function
(a generalized function)

1. $\boxed{\int_{-\infty}^{\infty} \delta(x)\, dx = 1}$

2. $\int_{-\epsilon}^{\epsilon} \delta(x)\, dx = 1$ if $\epsilon > 0$.

3. $\delta(x)$ does not really exist at $x = 0$.

4. $\delta(x) = 0$ if $x \neq 0$.

5. $\int_{-\infty}^{\infty} g(x)\, \delta(x - x_0)\, dx = g(x_0)$ $\forall x_0$

$\delta(x_0 - x)$

$\to g(x_0)$

for any continuous $g(x)$.

Given $\epsilon > 0$,

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = \int_{x_0-\epsilon/2}^{x_0+\epsilon/2} g(x)\,\underbrace{\delta(x-x_0)}\,dx$$

$$d_\epsilon(x) = \frac{1}{\epsilon}$$

$$\xrightarrow{\epsilon \to 0} \simeq \underset{\smile}{\epsilon}\, g(x_0)\, \underset{\sim}{\frac{1}{\epsilon}} = g(x_0)$$

6

$$\int_{-\infty}^{x} \delta(v)\,dv = M(x) \Rightarrow \frac{dM(x)}{dx} = \delta(x)$$

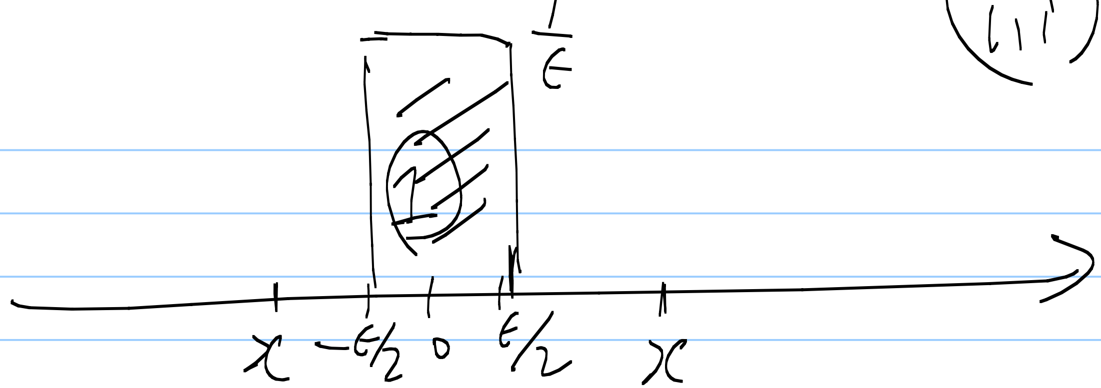
$x \neq 0$

i if $x < 0$, $M(x) = 0$

ii if $x > 0$, $M(x) = 1$

$$d_\epsilon(x) \xrightarrow{\epsilon \to 0} \delta(x)$$

(iii) if $x=0$, ?

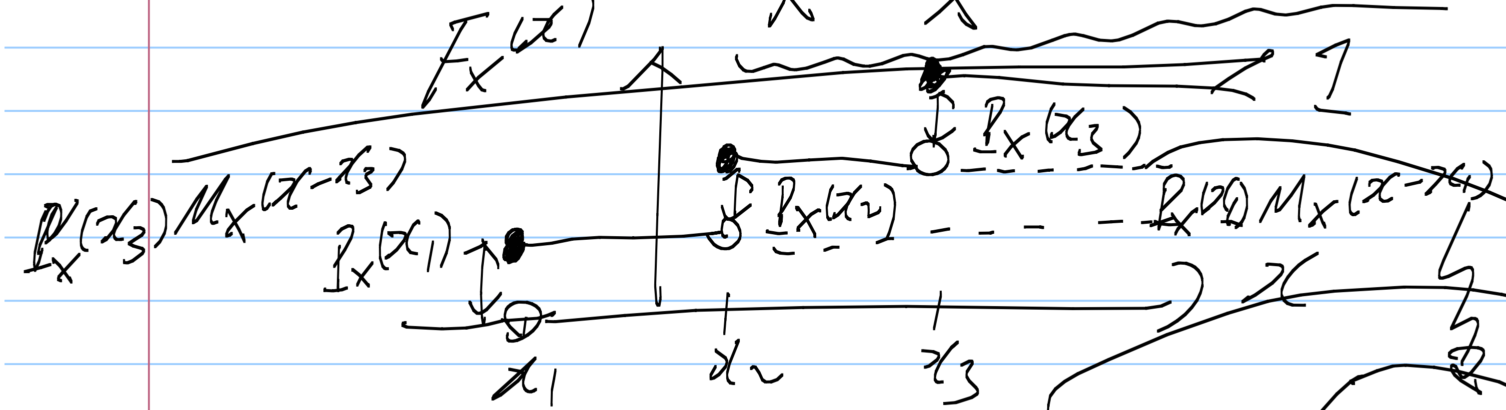


(7) $\delta(x) = \dfrac{dM(x)}{dx}$ $\quad x \neq 0$

The derivative of $M(x)$ does not really exist at $x=0$.

Consider a PMF $P_X(x)$ for a discrete random variable X. Then, its CDF

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, u(x - x_i) \quad \checkmark$$



$$\frac{du(x)}{dx} = \delta(x)$$

$$\Rightarrow \frac{dF_X(x)}{dx} = f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\,\delta(x - x_i)$$

$$\int_{-\infty}^{x} f_X(v)\,dv = \sum_{x_i \in S_X} P_X(x_i) \int_{-\infty}^{x} \delta(v - x_i)\,dv \;, \quad x \neq x_i,\ \forall i$$

$$= \sum_{x_i \in S_X,\ x_i < x} P_X(x_i)$$

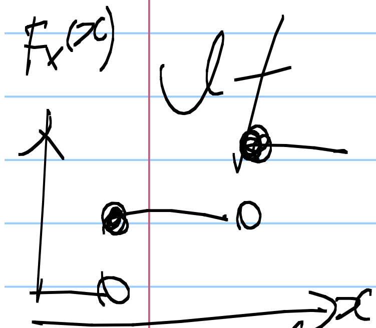
For any random variable X, its CDF

$$F_X(x) \triangleq \mathbb{P}[X \le x]$$

$F_X(-\infty) = 0$ and $F_X(+\infty) = 1$.

is well-defined. It is nondecreasing and continuous from the right with

If $F_X(x)$ is continuous, then X is a continuous random variable.



If $F_X(x)$ is a staircase function, then X is a discrete random variable.

If $F_X(x)$ is neither a staircase nor a continuous function, then X is a mixed random variable.

$$f_Y(y) = \frac{2}{9}\left[u(y) - u(y-3)\right] \quad \le$$

$$+ \frac{1}{3}\delta(y)$$

$$= \begin{cases} \frac{2}{9} + \frac{1}{3}\delta(y), & 0 \le y < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = 0 + \frac{2}{9}\int_0^3 y\, dy = 1$$

$$Y = g(X) = \begin{cases} 1, & X \le 0 \\ 3, & X > 0 \end{cases}$$

$$F_Y(y) = P[Y \le y]$$

(i) $y < 1$, $F_Y(y) = 0$

(ii) $y > 3$, $F_Y(y) = 1$

(iii) $1 \le y < 3$, $F_Y(y) = P[Y \le y] = P[Y = 1]$

$$= P[X \le 0] = F_X(0)$$

(iv) $y = 3$, $F_Y(y) = P[Y \le 3] = P[Y = 1] + P[Y = 3]$

$$= P[X \le 0] + P[X > 0] = 1$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 1 \\ 1 & y \ge 3 \\ F_X(0) & 1 \le y < 3 \end{cases}$$

$$\Rightarrow F_Y(y) = \underline{u(y-3)} + F_X(0)\left[u(y-1) - u(y-3)\right]$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = \delta(y-3) + F_X(0)\left[\delta(y-1) - \delta(y-3)\right]$$

$*$

$$W = \begin{cases} -10 & V < -10 \\ V & -10 \le V \le 10 \\ 10 & V > 10 \end{cases}$$

$$F_W(w) = P[W \le w]$$

(i) if $w < -10$, $F_W(w) = 0$

$$u(x) = \begin{cases} 1 & x \ge 0 \end{cases}$$

(ii) if $w \ge 10$, $F_W(w) = 1$

(iii) if $-10 \le w < 10$,

$$F_W(w) = P[W \le w] = P[V \le w] = F_V(w)$$

$$\Rightarrow F_W(w) = \begin{cases} 0 & w < -10 \\ F_V(w) & -10 \le w < 10 \\ 1 & w \ge 10 \end{cases}$$

$$\Rightarrow F_W(w) = F_V(w)\left[u(w+10) - u(w-10)\right] + u(w-10)$$

$$\Rightarrow f_W(w) = \frac{d}{dw} F_W(w)$$

$$= f_V(w)\left[u(w+10) - u(w-10)\right] + F_V(w)\left[\delta(w+10) - \delta(w-10)\right] + \delta(w-10)$$

$$= f_V(w)\left[u(w+10) - u(w-10)\right] + F_V(-10)\,\delta(w+10) - F_V(10)\,\delta(w-10) + \delta(w-10)$$

$$\rightarrow (1 - F_V(10))\,\delta(w-10)$$

where $\left(F_V(v)\right) = \Phi\left(\frac{v-\mu_V}{\sigma_V}\right) = \underline{\Phi\left(\frac{v}{5}\right)}$.

Ex: $Y = X^2$ and X is uniform in $[-1, 3]$.

Sol: $F_Y(y) \stackrel{\Delta}{=} P[Y \le y]$.

(i) If $y < 0$, $F_Y(y) = 0$

(ii) If $0 \le y < 1$, $F_Y(y) = P[Y \le y] = P[X^2 \le y]$

$= P[-\sqrt{y} \le X \le \sqrt{y}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

(iii) If $1 \le y$, $F_Y(y) = P[Y \le y] = P[X^2 \le y]$

$= P[-1 \le X \le \sqrt{y}] = F_X(-1) - F_X(\sqrt{y})$

$$\Rightarrow \bar{F}_Y(y) = \begin{cases} 0 & y<0 \\ \bar{F}_X(-\sqrt{y}) - \bar{F}_X(\sqrt{y}) & 0 \le y < 1 \\ \bar{F}_X(-1) - \bar{F}_X(\sqrt{y}) & 1 \le y \end{cases}$$

$$= \left(\bar{F}_X(-\sqrt{y}) - \bar{F}_X(\sqrt{y})\right)\left(u(y) - u(y-1)\right)$$

$$+ \left(\bar{F}_X(-1) - \bar{F}_X(\sqrt{y})\right) u(y-1)$$

1

$$\Rightarrow f_Y(y) = \left(f_X(-\sqrt{y})\frac{-\frac{1}{2}}{\sqrt{y}} - f_X(\sqrt{y})\frac{\frac{1}{2}}{\sqrt{y}}\right)\left(u(y) - u(y-1)\right)$$

$$+ \left(\bar{F}_X(-\sqrt{y}) - \bar{F}_X(\sqrt{y})\right)\left(\delta(y) - \cancel{\delta(y-1)}\right)$$

2

$$+ f_X(\sqrt{y})\frac{\frac{1}{2}}{\sqrt{y}}\, u(y-1)$$

3

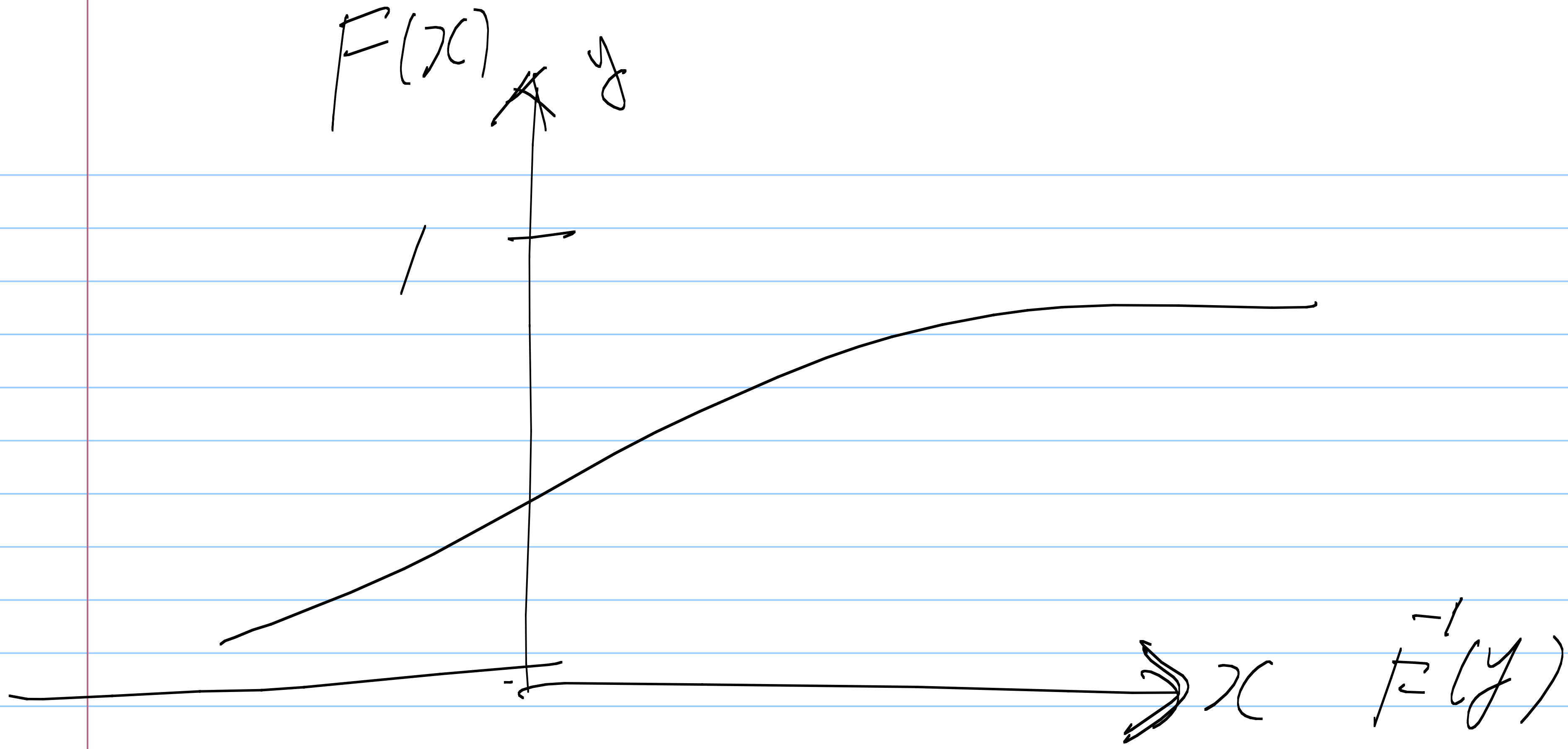
$$+ \left(\bar{F}_X(-1) - \bar{F}_X(\sqrt{y})\right)\cancel{\delta(y-1)}$$

Where $F_X(x) = P[X \le x]$

$$= \begin{cases} 0 & x < -1 \\ 1 & x \ge 3 \\ \frac{x+1}{4} & -1 \le x < 3 \end{cases}$$

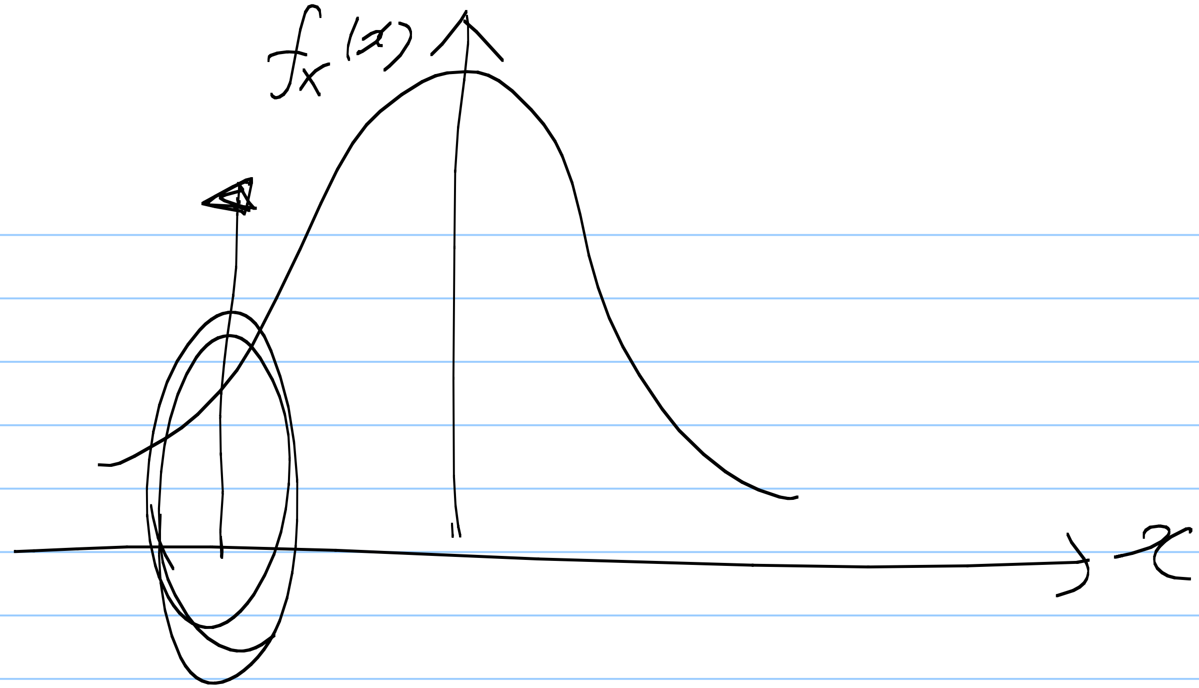
$$\Rightarrow F_X(\underset{\sqrt{y}}{x}) = \frac{x+1}{4}\left[u(x+1) - u(x-3)\right] + u(x-3)$$

$$u(\sqrt{y}+1) = \begin{cases} 1 & \sqrt{y} \ge -1 = 1 \\ 0 & \sqrt{y}+1 < 0 \end{cases}$$



$$u(\sqrt{y}-3) = \begin{cases} 1 & \sqrt{y} \geq 3 \\ 0 & \sqrt{y} < 3 \end{cases}$$

$$= \begin{cases} 1 & y \geq 9 \\ 0 & y < 9 \end{cases} = u(y-9)$$



$f_X(x)$
x

✓ $F_X(x) = \left(x - \frac{x^2}{4}\right)\left(u(x) - u(x-2)\right) + u(x-2)$

$$Y = \begin{cases} X, & X \le 1 \\ 1, & X > 1 \end{cases}$$

✓ $F_Y(y) = P[Y \le y]$

(i) if $1 \le y$, $F_Y(y) = 1$

(ii) if $y < 1$, $F_Y(y) = P[Y \le y] = P[X \le y] = F_X(y)$.

$\Rightarrow F_Y(y) = \begin{cases} 1, & 1 \le y \\ F_X(y), & y < 1 \end{cases} = F_X(y)\left[1 - u(y-1)\right] + u(y-1)$

$\Rightarrow f_Y(y) = f_X(y)\left[1 - u(y-1)\right] - F_X(y)\,\delta(y-1) + \delta(y-1)$

$$= \underline{f_X(y)[1-u(y-1)] + (1-F_X(y))\,\delta(y-1)} \quad \#$$

- $\underline{f_{X|B}}(x) \triangleq \lim_{dx \to 0} P\left[X \in [x, x+dx) \,\middle|\, X \in B\right] \frac{1}{dx}$ is called the <u>conditional PDF</u> of X given event B. Note: $f_{X|B}(x)\,dx = P[X \in [x, x+dx) | X \in B]$

$f_{X|B}(x)$ is itself a PDF. It satisfies the same properties as $f_X(x)$. For example,

① $f_{X|B}(x) \geq 0, \quad \forall x$

(2) $\int_{-\infty}^{\infty} f_{X|B}(x)\,dx = 1$

(3) There exists one $F_{X|B}(x) = \int_{-\infty}^{x} f_{X|B}(y)\,dy$ such that $F_{X|B}(x)$ is a CDF.

$B \subset S_X$

$$f_{X|B_i}(x) \stackrel{\text{def}}{=} \lim_{dx\to 0} P[\underbrace{X \in [x, x+dx)}_{A} \mid \underbrace{X \in B_i}_{B_i}] \frac{1}{dx}$$

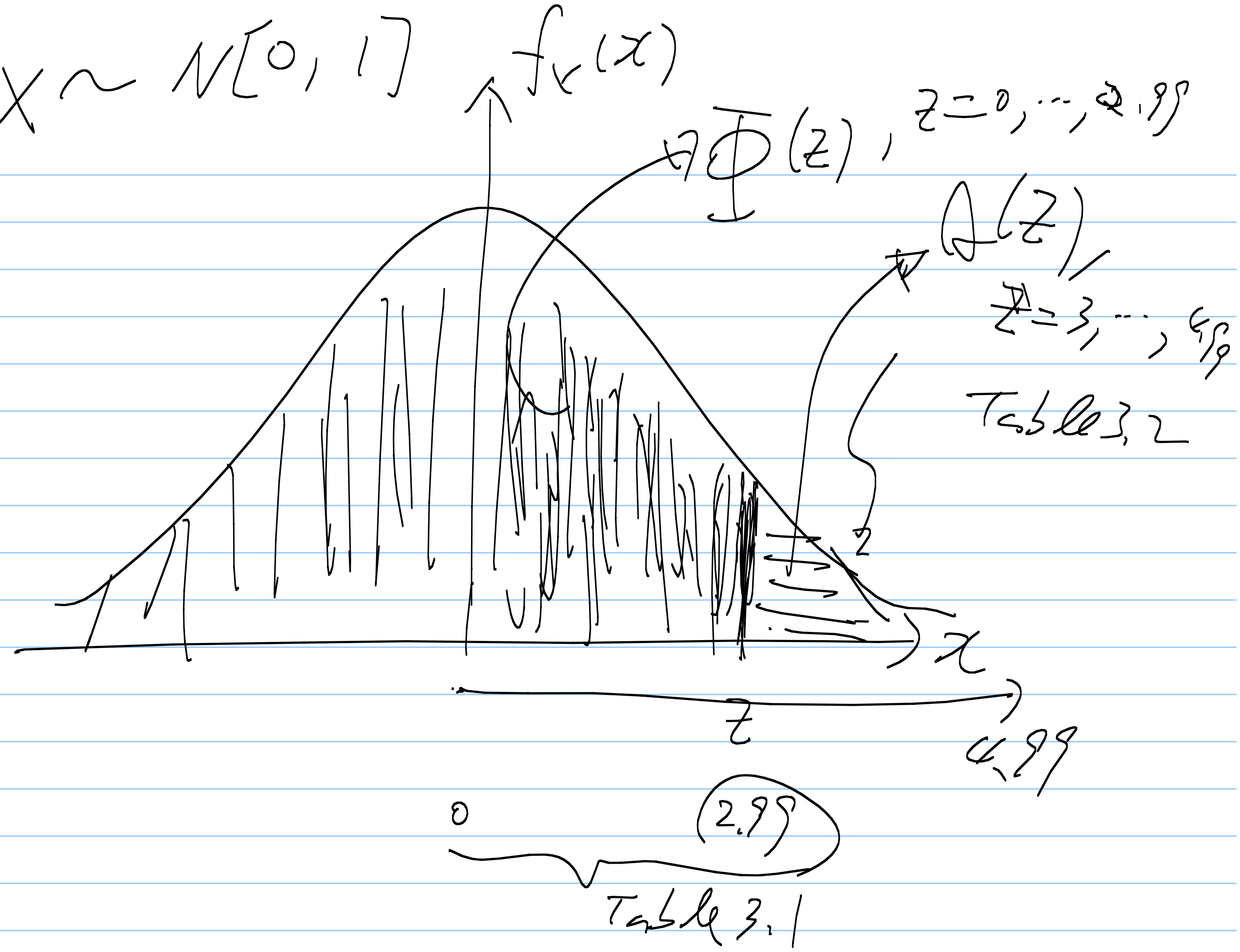
$$P[\underbrace{X \in [x, x+dx)}_{A}] = \sum_i P[\underbrace{X \in [x, x+dx)}_{A} \mid B_i] P[B_i]$$

from the law of total probability

Dividing by dx and taking $\lim_{dx\to 0}$ gives

$$f_X(x) = \sum_i f_{X|B_i}(x) P[B_i].$$

$X \sim N[0, 1]$



$f_X(x)$
$\Phi(z)$, $z = 0, \dots, 2.99$
$Q(z)$, $z = 3, \dots, 4.99$
Table 3.2
x
z
4.99
0
2.99
Table 3.1

For a discrete random variable X,

$$\underbrace{f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\,\delta(x - x_i)}$$

PDF

$$\Rightarrow E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \underbrace{\sum_{x_i \in S_X} P_X(x_i)\,\delta(x - x_i)}_{= g(x)}\, dx$$

$$= \sum_{x_i \in S_X} P_X(x_i) \underbrace{\int_{-\infty}^{\infty} x\,\delta(x - x_i)\, dx}_{= g(x_i) = x_i}$$

$$= \sum_{x_i \in S_X} x_i\, P_X(x_i)$$

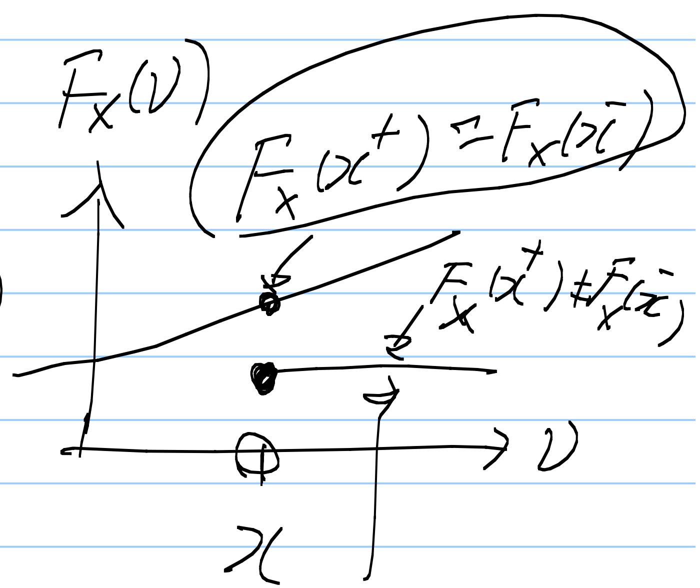
#

$$x^+ \triangleq \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} (x+\epsilon)$$

$$x^- \triangleq \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} (x-\epsilon)$$

$$F_X(x^+) \triangleq \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} F_X(x+\epsilon)$$

$$F_X(x^-) \triangleq \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} F_X(x-\epsilon)$$



$F_X(x^+) = F_X(x)$

$F_X(x^-) \neq F_X(x)$