Chap 1:

- This textbook provides
  - The logic of probability theory
  - Applications of probability theory ~~to~~ practical problems
  - statistical inference
- Three categories of theoretical material

  Definitions: establishing the logic

  Axioms: facts without proof

  Theorems: consequences following from definitions and axioms

- Set : A collection of elements
- $x \in A$ means "x is an element of set A."

  $x \notin A$ means "x is not an element of set A."

  $\in$ denotes set inclusion.
- Ways to define a set :

  $A = \{$National Taiwan University, National Chiao Tung University$\}$

  $B = \{$All NTU juniors who weight more than 60 kg$\}$

  $C = \{x^2 \mid x = 1, 2, 3, 4, 5\} = \{1, 4, 9, 16, 25\}$

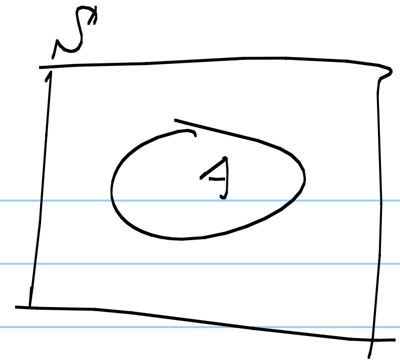
  $D = \{x^2 \mid x = 1, 2, 3, \cdots\}$

- Subset : $A \subset B$ if and only if ($\Leftrightarrow$)
  A is a subset of B iff "if $x \in A$, then $x \in B$."
- Set Equality : $A = B$
  A and B are identical iff "$A \subset B$ and $B \subset A$."
  A set is unaffected by the order of the elements, e.g., $\{1, 2, 3\} = \{3, 1, 2\}$.
- Universal Set : S
  S includes all of the elements of all the sets in the study.
  For any set X, $X \subset S$.
- Null Set : ϕ
  ϕ includes no elements.
  For any set X, $\phi \subset S$.

- Venn Diagrams

  Large Rectangle denotes S

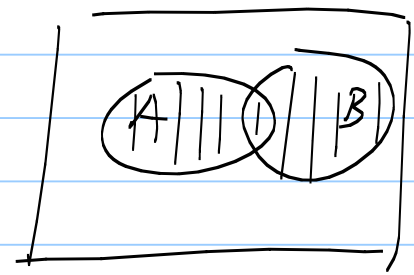
  Closed Surfaces denote sets.



- Set Operations and Venn Diagrams

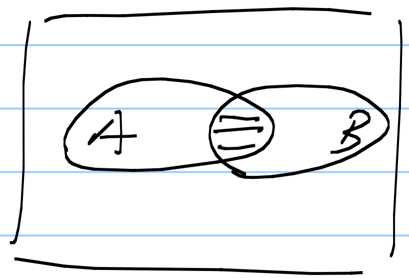
  * Union: $A \cup B$

    $x \in A \cup B$ iff "$x \in A$ or $x \in B$"



  * Intersection: $A \cap B$ or AB

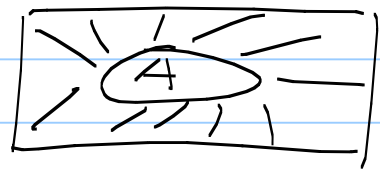
    $x \in A \cap B$ iff "$x \in A$ and $x \in B$"



  * Complement: A^c

    $x \in A^c$ iff "$x \notin A$"

    Note: $S^c = \phi$



* Difference : $A - B$

$x \in A - B$ iff "$x \in A$ and $x \notin B$."

Notes: $A - B = A \cap B^c$, $A^c = S - A$, $A - B \subset A$

- Defns: ① $A_1, A_2, \cdots, A_n$ are mutually exclusive iff

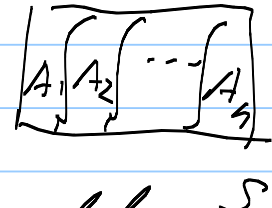
$$A_i \cap A_j = \phi \quad \text{for all } i \neq j$$

② A and B are disjoint iff

$$A \cap B = \phi$$

③ $A_1, A_2, \cdots, A_n$ are collectively exhaustive iff

$$A_1 \cup A_2 \cup \cdots \cup A_n \equiv \bigcup_{i=1}^{n} A_i = S$$

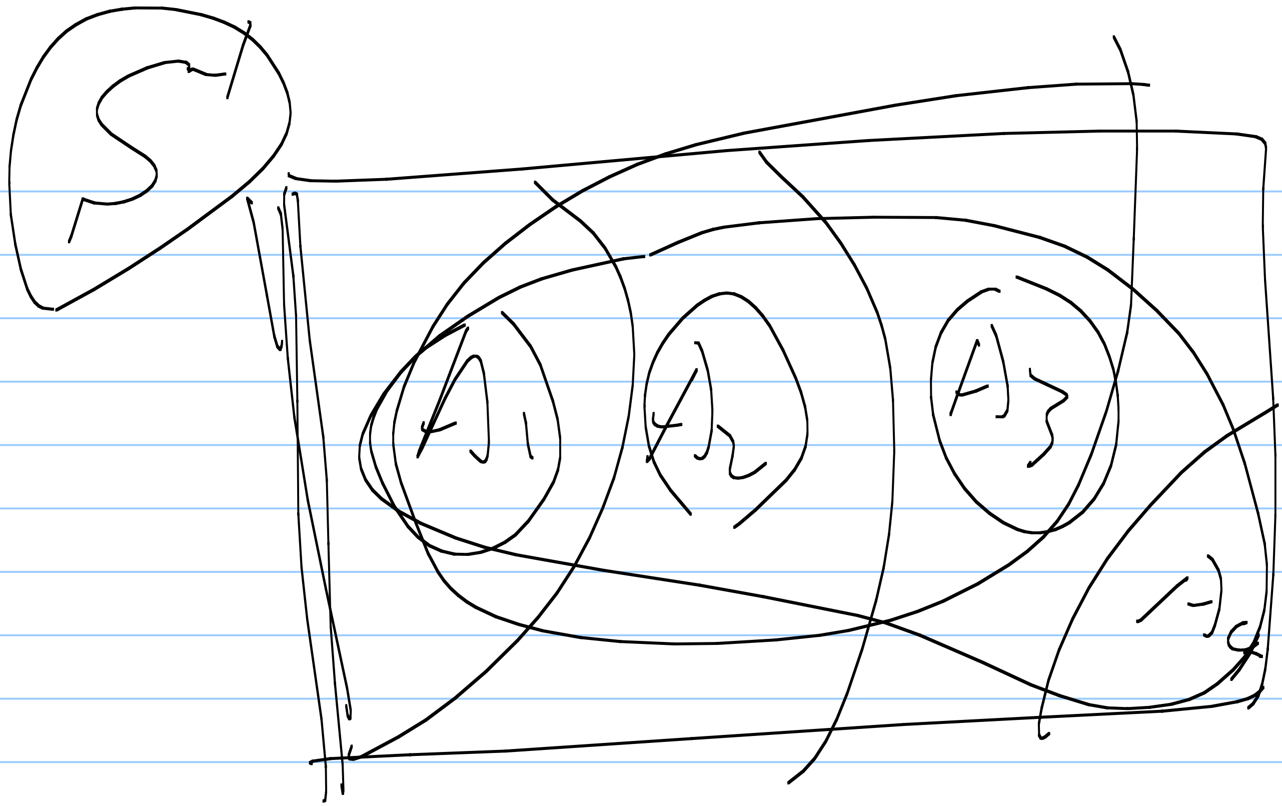


Note: $A_1, A_2, \cdots, A_n$ are commonly said to "partition" S if they are both mutually exclusive and collectively exhaustive.

- Probability is a number that describes a set.
- Repeatable Experiment With Uncertain Outputs:

Such an experiment consists of a procedure and observations.

We desire to analyze models of actual physical experiments with uncertain outputs.



$P[A_i] \geq 0 \qquad P[S] = 1$

Notes: ① $P[\phi] = 0$

Pf: Prove by contradiction.

Suppose $P[\phi] > 0$.

According to Axiom 3, if $A_1 = A_2 = \cdots = \phi$, then $P[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} P[A_i]$

$$= \sum_{i=1}^{\infty} P[\phi]$$

Since $\bigcup_{i=1}^{\infty} A_i = \phi$, $P[\phi] = \sum_{i=1}^{\infty} P[\phi]$ which is impossible if $P[\phi] > 0$. Thus, $P[\phi] = 0$

Q.E.D.

② If $P[A] = 0$, then A is NOT necessarily a null set. Try to come out with a counter example.

③ For any event A, $P[A] \le 1$.

pf: According to Axiom 3, if $A_1 = A$, $A_2 = A^c$, $A_3 = A_4 = \cdots = \phi$, then

$$P\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P[A_i]$$

$$\Rightarrow P[A \cup A^c] = P[A] + P[A^c] \quad \text{(Note ①)}$$

$$\Rightarrow \underbrace{P[S] = 1}_{\text{Axiom 2}} = P[A] + \underbrace{P[A^c]}_{\ge 0 \text{ Axiom 1}} \ge P[A]$$

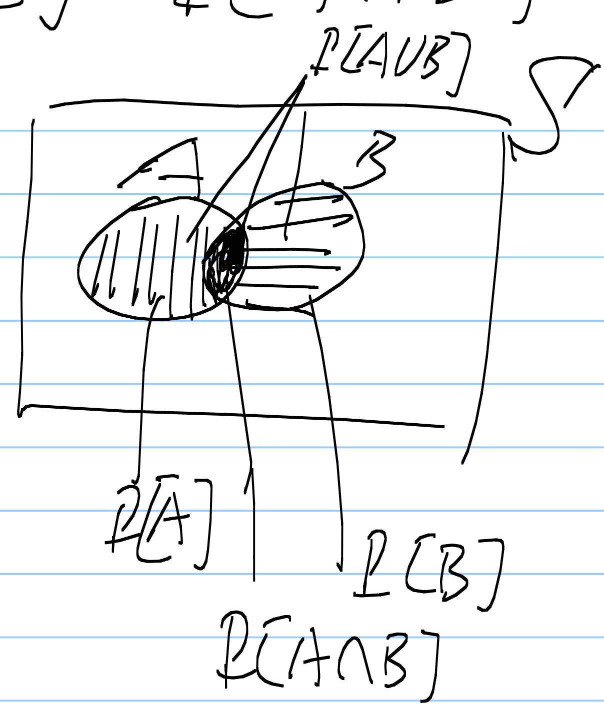
(4) $P[A^c] = 1 - P[A]$.   Q.E.D.

For events A and B,

$$\mathbb{P}[\underbrace{A \cap B}_{\text{joint event}}] = \mathbb{P}[AB] = \mathbb{P}[A, B]$$

and

Thm : $P[A \cup B] = \underbrace{P[A] + P[B]} - P[A \cap B]$



Pf :

① Because $A \cap \widetilde{B-A} = \phi$

and $A \cup B-A = A \cup B$,

$P[A \cup B] = P[A] + \underbrace{P(B-A]}$

according to Axiom 3.

② Because $B \cap A \cap B-A = \phi$

and $B \cap A \cup B-A = B$,

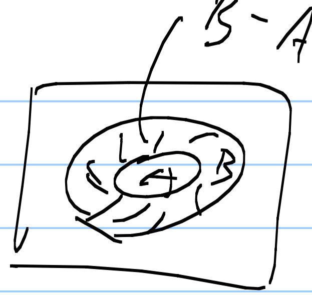
$P[B] = P[B \cap A] + P[B-A] \Rightarrow P[B-A] = \underbrace{P[B] - P[A \cap B]}$

according to Axiom 3.

③ ① + ② $\Rightarrow P[A \cup B] = P[A] + P[B] - P[A \cap B]$.

Q.E.D.

(d) If $A \subset B$, then $P[A] \leq P[B]$.



Pf: Because $A \cap B-A = \phi$

and $\underbrace{A \cup B-A = B}$ ($\because A \subset B$)

$P[B] = P[A] + P[B-A] \geq P[A]$

according to Axiom 3, Axiom 1

Thm 1.8: $P[A] = \sum_{i=1}^{m} P[A \cap B_i]$ (law of total probability)

if $[B_1, B_2, \cdots, B_m]$ is an event space.

Pf: Because $A = \bigcup_{i=1}^{m} \{A \cap B_i\}$ and

$\{A \cap B_i\} \cap \{A \cap B_j\} = \phi,\ i \neq j$,

$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$

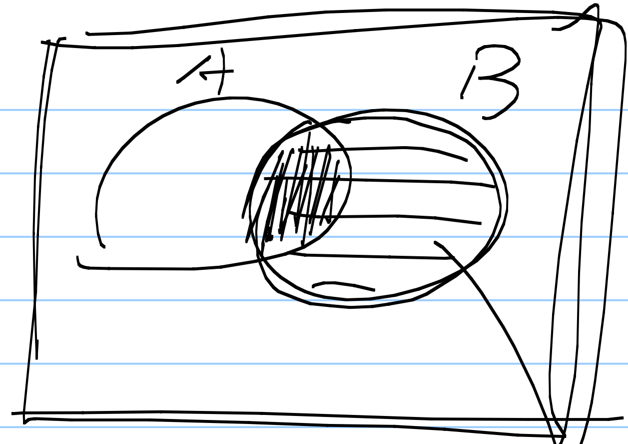
according to Axiom 3.

$P[A] \approx 1$, $P[A] = 0$, $P[A] = \frac{1}{2}$

$P[A] \equiv$ *a priori* (or prior) probability of event A

unconditional probability of event A

$P[A|B]$ = conditional probability of event A given event B



$$\frac{P[AB]}{P[B]}$$

new universe

$P[B] > 0.$

$$\underbrace{P[AB]}_{\text{joint probability}} = \underbrace{P[A]}_{\text{marginal probability}}\, \underbrace{P[B]}$$

$$\Rightarrow \quad P[A|B] = \frac{P[AB]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$$

(if $P[B] > 0$)

if A and B are independent

Similarly, if $P[A] > 0$,

$P[B|A] = P[B]$ provided that A and B are independent.

A_1, A_2, A_3 are called (mutually) independent iff $\underline{P[A_i A_j] = P[A_i]\,P[A_j],\ i \neq j}$, and $P[A_1 A_2 A_3] = \prod_{i=1}^{3} P[A_i]$.

A_1, A_2, A_3 are independent in pairs or pairwise independent iff $P[A_i A_j] = P[A_i]\,P[A_j]$ for $i \neq j$.

Note: Mutually Independent $\underset{\not\Leftarrow}{\Rightarrow}$ Independent in pairs

$A_1, A_2, \cdots, A_n$, with $n \geq 2$, are called (mutually) independent iff $\mathbb{P}\left[\bigcap_{i=1}^{m} B_i\right] = \prod_{i=1}^{m} \mathbb{P}[B_i]$ where $B_1, B_2, \cdots, B_m$ are m distinct events obtained from $A_1, A_2, \cdots, A_n$, for all $2 \leq m \leq n$.

Note: $2^n - (n+1)$ equalities are required to establish that n events are independent.

Sample

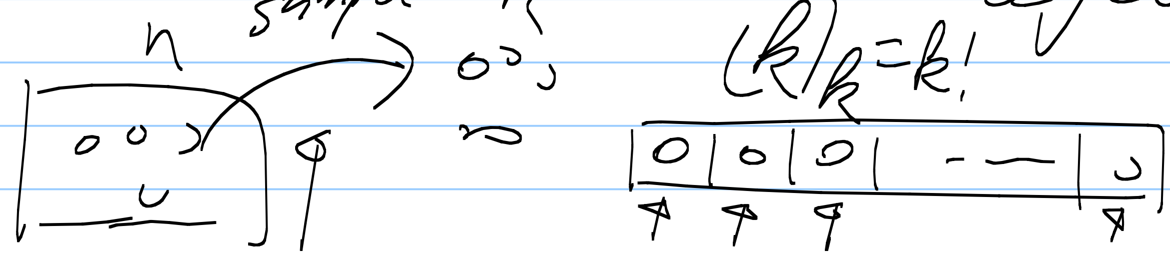
Sampling without replacement

with

$\binom{n}{k} \equiv k$-combination

$\equiv n$-choose-k

$(n)_k \equiv k$-permutation of n distinguished objects



$(k)_k = k!$

$$\underbrace{\binom{n}{k}}\text{\_\_\_\_\_\_\_\_\_\_\_\_} . \qquad \underbrace{k!} = \underbrace{(n)_k} = \frac{n!}{(n-k)!}$$

$$\Longrightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!} = C^n_k$$

combinction

n subexperiments with $S = \{s_0, s_1, \cdots, s_{m-1}\}$

s_0 ---- n_0

s_1 ---- n_1

$\vdots$

s_{m-1} ---- n_{m-1}

$$\sum_{i=0}^{m-1} n_i = n$$

binomial expansion

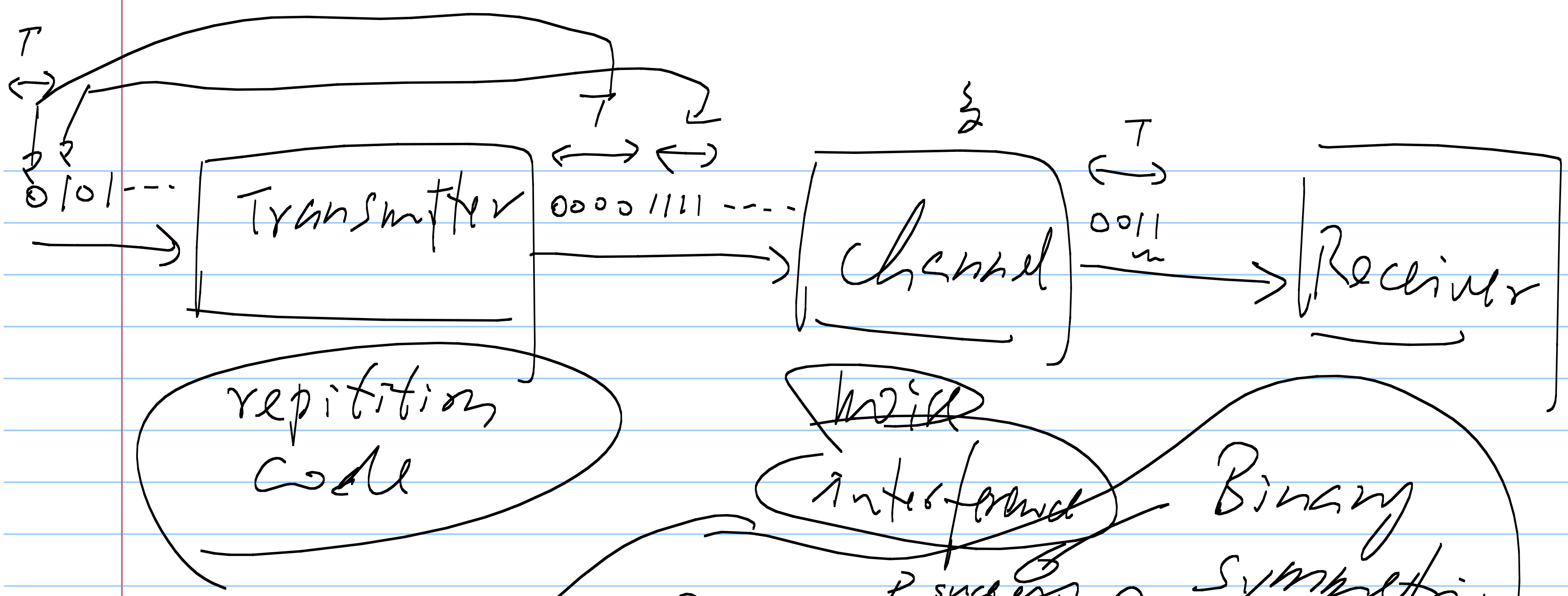
$$(x+y)^n = \sum_{n_0=0}^{n} \binom{n}{n_0} x^{n_0} y^{n-n_0}$$

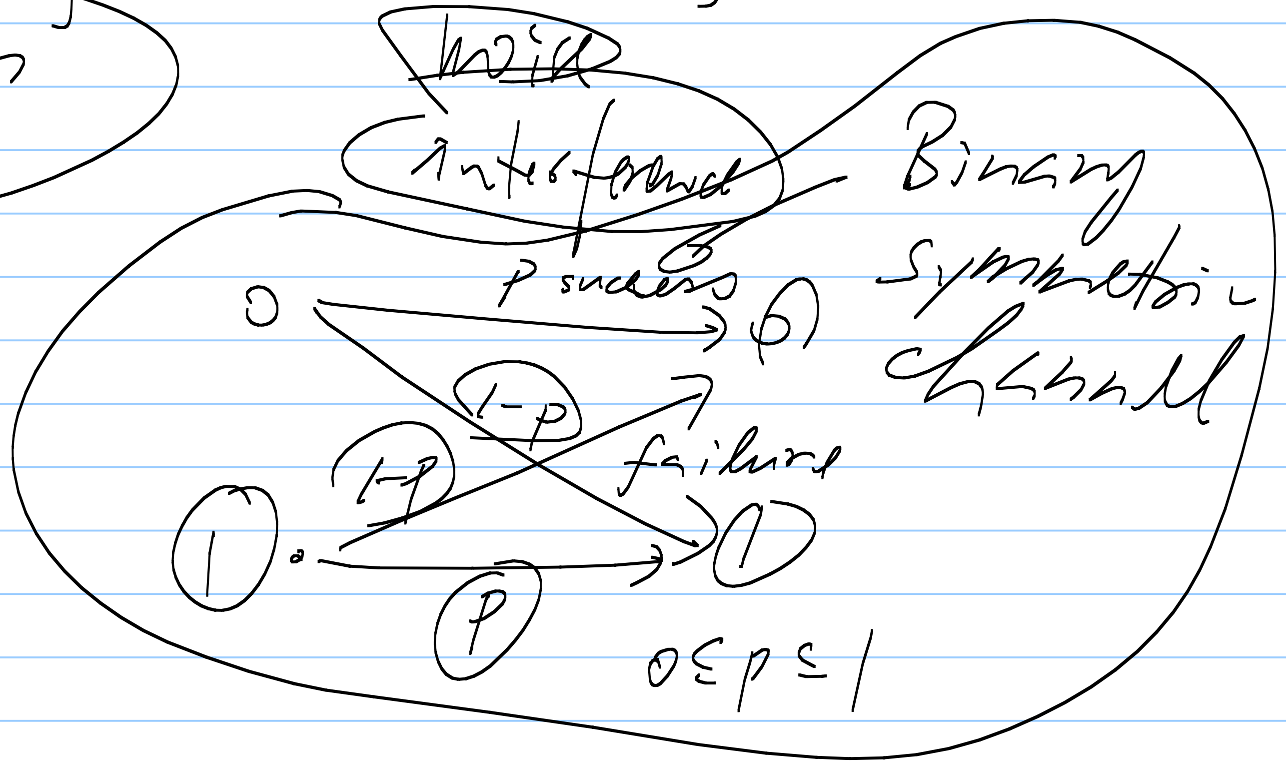
$$\binom{n}{n_0, n_1, \cdots, n_{m-1}} \equiv \text{multinomial coefficients}$$

$$\binom{n}{n_0} \equiv \text{binomial coefficient}$$

$$\left(\sum_{i=0}^{m-1} x_i\right)^n = \sum_{\substack{n_0, n_1, \ldots, n_{m-1} \in \{0,1,\ldots,n\} \\ n_0 + n_1 + \cdots + n_{m-1} = n}} \binom{n}{n_0, n_1, \ldots, n_{m-1}} \prod_{i=0}^{m-1} x_i^{n_i}$$

multinomial expansion


T
0101 ...
Transmitter
T
00001111 ....
repitition code
Channel
ξ
~~noise~~
interference
T
0011
Receiver


Binary symmetric channel
0
p success
0
1-p
1-p
failure
1
p
1
0 ≤ p ≤ 1

$q = 1 - p$
$= 0.1$

independent repetitions of a subexperiment

S for each subexperiment is defined as $\{\underset{p}{\text{success}}, \underset{1-p}{\text{failure}}\}$

$S_{n_0, n_1} \equiv$ the event that n_0 failures and n_1 successes occur in $n = n_0 + n_1$ trials (~~or subexperiments~~)

Consider n independent trials.
Each trial has the identical sample space $S = \{s_0, s_1, \cdots, s_{m-1}\}$.

$S_{n_0, n_1, \cdots, n_{m-1}}$ $\equiv$ the event that

s_0 occurs n_0 times

s_1 occurs n_1 times

$\vdots$

s_{m-1} occurs n_{m-1} times

in $n = \sum_{i=0}^{m-1} n_i$ trials.

$$P[\quad] = \binom{n}{n_0, n_1, \cdots, n_{m-1}} \prod_{i=0}^{m-1} p_i^{n_i}$$

Where p_i represents the probability of event $\{s_i\}$ in a single trial.