1. (5%; 1% each) Let X and Y be independent and identically distributed Gaussian random variables which have zero mean and unit variance. Determine whether each of the following statements is true or false. Correct answer without any explanation or proof will not be given any credit.

(a) If a is a constant with $0 < a < 1$, then $aX + (1-a)Y$ is a Gaussian random variable with zero mean and unit variance.

Sol: From Theorem 3.19, aX and $(1-a)Y$ are independent Gaussian random variables with zero mean and variance a^2 and $(1-a)^2$, respectively. Moreover, from Theorem 6.10, $aX + (1-a)Y$ is a Gaussian random variable with zero mean and variance $a^2 + (1-a)^2$. This statement is false.

(b) $E[X^n Y^m] = 0$ if $n + m$ is an odd integer.

Sol: Without loss of generality, we assume n is an even integer and m is an odd integer. Because X and Y are independent, we have $E[X^n Y^m] = E[X^n]E[Y^m]$. Furthermore, y^m is an odd funtion and $f(y)$ is a even function, so $E[Y^m] = 0$. As a consequence, $E[X^n Y^m] = 0$, and this statement is true.

(c) $E[X|Y] = 0$.

Sol: Since X and Y are independent, $E[X|Y] = E[X] = 0$. This statement is true.

(d) $E[\exp(X+Y)] = 1/e$ where e is the exponential.

Sol: $E[\exp(X+Y)] = E[\exp(X)]E[\exp(Y)] = \left\{ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(x)\exp(-\frac{x^2}{2}) \right\}^2 =$

$\left\{ \sqrt{e} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{(x-1)^2}{2}) \right\}^2 = e$, so this statement is false.

(e) The probability density function of the random variable $X + Y$ is an even function.

Sol: From Theorem 6.10, the random variable $W = X + Y$ is a Gaussian random variable with zero mean and variance 2, so the PDF of W is an even function. This statement is true.