

CHAI 6

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Homework # 5:

page: 238: 5.1.3, 5.2.2, 5.3.4

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Done on May 29, at 5:20pm.

By Taylor series expansion,

$$\Phi_X(s) = \mathbb{E}[e^{sX}] = \mathbb{E}\left[\sum_{n=0}^{\infty} \frac{(sX)^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{s^n}{n!} \mathbb{E}[X^n]$$

$$\frac{d^n \Phi_X(s)}{ds^n} \Big|_{s=0}$$

Def: $\Phi_X(s) = \mathbb{E}[e^{sX}]$ is called the characteristic function (CF) of X .

Note: The MGF $\Phi_X(s)$ may exist only for

certain region of s , i.e., the region of convergence.

② The CF $\Phi_X(\omega)$ exists for all ω values

$$\text{Since } |\Phi_X(\omega)| = \left| \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx \right|$$

$$\leq \int_{-\infty}^{\infty} \underbrace{|f_X(x)| e^{j\omega x}}_{\text{"}|f_X(x)|"} dx$$

$$\text{③ } \Phi_X(-\omega) = \int_{-\infty}^{\infty} f_X(x) e^{-j\omega x} dx \stackrel{\text{"}f_X(x)\text{"}}{=} \text{the Fourier transform of } f_X(x).$$

Proof of Central Limit Theorem (CLT):

CLT: When X_n 's are iid with identical mean μ_x and identical variance σ_x^2 ,
we define the new random variables

$$W_n \equiv \frac{X_n - \mu_x}{\sigma_x}, \quad A_n, \quad \text{and}$$

$$Y_n \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^n W_i.$$

and obtain according to CLT that

$$\bar{F}_{Y_n}(y) = \Phi(y). \quad \square$$

proof: let us check the moment generating function of Y_n :

$$\phi_{Y_n}(s) \equiv E[\exp(sY_n)]$$

$$= E\left[\exp\left(s \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i\right)\right]$$

X_i are

iid with

$$\phi = \prod_{i=1}^n E\left[\exp\left(\frac{s}{\sqrt{n}} X_i\right)\right]$$

mean = 0 and var = 1

$$\text{and MGF } \phi_M(s) = \underbrace{\left[\phi_M\left(\frac{s}{\sqrt{n}}\right)\right]^n} \dots \textcircled{*} \text{ or}$$

Expanding $\phi_M\left(\frac{s}{\sqrt{n}}\right)$ as a Taylor series, we

have

$$\phi_M\left(\frac{s}{\sqrt{n}}\right) = E\left[\exp\left(\frac{s}{\sqrt{n}} X\right)\right]$$

$$P = \sum_{n=6}^{\infty} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(\frac{s}{h} u \right)^n \right]$$

$$= 1 + \frac{s}{h} \underbrace{E[u]}_{=0} + \frac{s^2}{2h} \underbrace{E[u^2]}_{=0} + \frac{s^3}{6h^{3/2}} \underbrace{E[u^3]}_{=0} + \dots$$

$$= 1 + \frac{s^2}{2h} + \underbrace{\left(\frac{1}{h} R(s, h) \right)}_{=0}$$

Where $R(s, h) = \frac{s^3}{6h^{1/2}} E[u^3] + \frac{s^4}{24h} E[u^4] + \dots$

and $\lim_{h \rightarrow \infty} R(s, h) = 0$,

Next, substituting the above series into (*)

gives $\phi_Y(s) = \left[1 + \frac{s^2}{2h} + \frac{1}{h} R(s, h) \right]^h$

Taking the natural logarithm further gives

$$\ln \phi_n(s) = n \ln \left[1 + \left(\frac{s^2}{2n} + \frac{1}{n} R(s, n) \right) \right]$$

For small $|x|$, i.e., $|x| \ll 1$,

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

which expands $\ln \phi_n(s)$ into

$$\ln \phi_n(s) = n \left[\left(\frac{s^2}{2n} + \frac{1}{n} R(s, n) \right) - \frac{1}{2} \left(\frac{s^2}{2n} + \frac{1}{n} R(s, n) \right)^2 + \dots \right]$$

$$= \frac{S^2}{2} + R(S, y) + \frac{1}{n} \underbrace{\left[\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right]}_{\substack{\text{no } n \rightarrow \infty \\ \text{no } n \rightarrow \infty}}$$

Thus, $\lim_{n \rightarrow \infty} \Phi_n(S) = e^{\frac{1}{2} S^2}$ which is the moment generating function of the standard normal random variable.

$$\text{i.i.o.,} \quad \lim_{n \rightarrow \infty} \bar{F}_n(y) = \Phi(y). \quad \text{Q.E.D.}$$

Note: There are several versions of CLT.

For example, there exist versions for correlated sequences. For more examples,

read the reference

F. Billingsley, "Probability and
Measure," 2nd ed, Wiley, 1986
(Section 27).