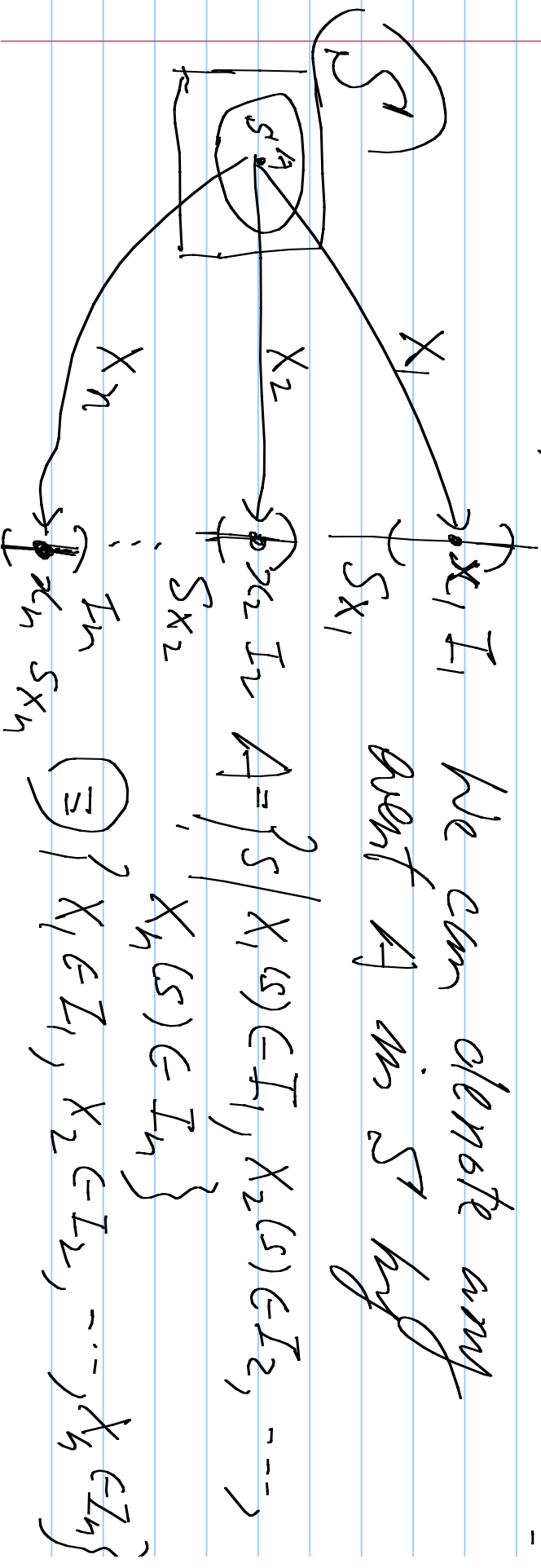


CHAP 5 Random Vectors

Defn: $\mathbf{X} \equiv [X_1, X_2, \dots, X_n]'$ where $'$ denotes transpose and X_1, X_2, \dots, X_n are random variables, is called an n -dimensional random vector.

Note: When $n=1$, \mathbf{X} becomes a random variable.



$$\textcircled{=} \{ \cancel{X} \in \mathbb{I} \}$$

where $\mathbb{I} \equiv \{ [x_1, x_2, \dots, x_n] \mid x_1 \in I_1, x_2 \in I_2, \dots, x_n \in I_n \}$

Then, $P[A] = P[\cancel{X} \in \mathbb{I}]$

Defn 5.3: Multivariate Joint PDF

The joint PDF of a continuous X_1, X_2, \dots, X_n is defined by

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{d^n F_{X_1, \dots, X_n}(x_1, \dots, x_n)}{dx_1 dx_2 \dots dx_n}$$

provided that $F_{X_1, \dots, X_n}(x_1, \dots, x_n)$ is differentiable.

$$f_{\underline{v}, \underline{w}}(\underline{v}, \underline{w}) = \begin{cases} \mathcal{C}, & 0 \leq v_1 \leq v_2 \leq 1, 0 \leq w_1 \leq w_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \underline{f_{\underline{v}}(\underline{v})} &= \int_{w_1}^1 \mathcal{C} \, dw_2 \, dw_1 = \int_0^1 \mathcal{C}(1-w_1) \, dw_1 = \mathcal{C}(w_1 - 2w_1^2) / 2 \\ \underline{f_{\underline{w}}(\underline{w})} &= \int_{v_1}^1 \mathcal{C} \, dv_2 \, dv_1 = \int_0^1 \mathcal{C} \, dv_2 \, dv_1 = \mathcal{C}v_2 \, dv_1 = \mathcal{C}v_1^2 / 2, \quad 0 \leq v_1 \leq v_2 \leq 1 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{f_{\underline{w}}(\underline{w})} &= \int_0^1 \int_{v_1}^1 \mathcal{C} \, dv_2 \, dv_1 & 0 \leq w_1 \leq w_2 \leq 1 \\ &= \mathcal{C} \int_0^1 (1-v_1) \, dv_1 = \mathcal{C}v_1 / 2, & 0 \leq w_1 \leq w_2 \leq 1 \\ &= 0, & \text{otherwise} \end{aligned}$$

Thus, $f_{\underline{v}, \underline{w}}(\underline{v}, \underline{w}) = f_{\underline{v}}(\underline{v}) f_{\underline{w}}(\underline{w})$. Thus,
 \underline{v} and \underline{w} are independent.

$$\underline{X}, \underline{W} = f(\underline{X})$$

$$(a) Y = \max\{X_1, X_2, \dots, X_n\}$$

$$F_Y(y) = P[Y \leq y] = P[\max\{X_1, X_2, \dots, X_n\} \leq y]$$

$$= P[X_1 \leq y, X_2 \leq y, \dots, X_n \leq y]$$

$$\stackrel{X_n\text{'s are}}{=} P\left[\prod_{i=1}^n P[X_i \leq y]\right] \quad (\text{independent})$$

$$= \prod_{i=1}^n F_X(y) \quad (\text{independently distributed})$$

$$= (F_X(y))^n$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy} = n(F_X(y))^{n-1} f_X(y) \quad \text{Q.E.D.}$$

(b) $M = \min\{X_1, X_2, \dots, X_n\}$

$$P[M > w] = P[\min\{X_1, X_2, \dots, X_n\} > w]$$

$$= P[X_1 > w, X_2 > w, \dots, X_n > w]$$

$$= \prod_{i=1}^n P[X_i > w] \quad (\text{independent})$$

$$= \prod_{i=1}^n (1 - F_X(w)) \quad (\text{independently distributed})$$

$$\Rightarrow F_M(w) = 1 - P[M > w] = 1 - (1 - F_X(w))^n$$

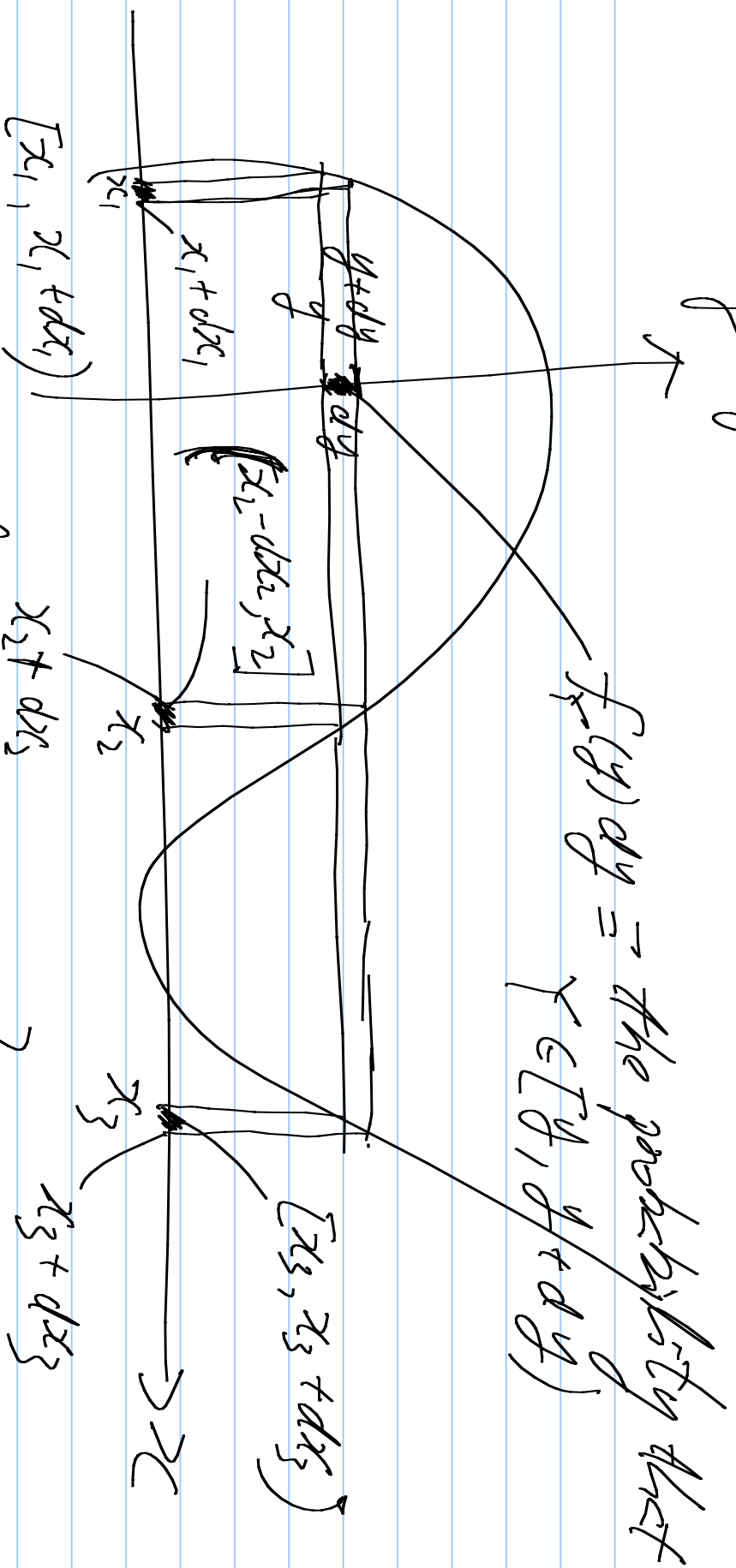
$$f_M(w) = \frac{dF_M(w)}{dw} = n(1 - F_X(w))^{n-1} f_X(w) \quad 0 \leq w$$

Consider $Y = g(X)$ where $y = g(x)$ is a well-defined function. For a given y , there exist $x_1(y), x_2(y), \dots, x_n(y)$ satisfying $y = g(x_i)$ $i=1, 2, \dots, n$. Then, $f_x(x)$ and $f_y(y)$ are related by

$$f_y(y) = \sum_{i=1}^n f_x(x_i(y)) \left| \frac{dg(x)}{dx} \right|_{x=x_i(y)}^{-1}$$

This is called the Jacobian rule of obtaining $f_y(y)$ from $f_x(x)$.

$$y = g(x)$$



Thus, the event $Y \in [y, y+dy]$

$$\Leftrightarrow \underbrace{X \in [x_1, x_1 + dx_1]} \cup \underbrace{X \in [x_2 - dx_2, x_2]} \cup \underbrace{X \in [x_3, x_3 + dx_3]}$$

$$\cup \{X \in [x_3, x_3 + dx_3]\}$$

$$\Rightarrow f_Y(y) dy = P[Y \in [y, y + dy]]$$

$$\begin{aligned} &\sim P[X \in [x_1, x_1 + dx_1]] + P[X \in [x_2 + dx_2, x_2]] \\ &\quad + P[X \in [x_3, x_3 + dx_3]] \end{aligned}$$

$$= f_X(x_1) dx_1 + f_X(x_2) dx_2 + f_X(x_3) dx_3$$

$$\Rightarrow f_Y(y) = \sum_{i=1}^3 f_X(x_i) \left| \frac{dx_i}{dy} \right|_{x=x_i}$$

Defn: X_1, X_2, \dots, X_n are called jointly Gaussian, or jointly normal, if any of their linear combination, i.e.,
 $\sum_{i=1}^n a_i X_i$ for any a_i 's, is normally

Gaussian,

Note: ① X_i is marginally Gaussian,

② Given X_1, X_2, \dots, X_n are marginally Gaussian, X_1, X_2, \dots, X_n may not be jointly Gaussian.

R_X and C_X satisfy the following properties:

① They are nonnegative definite, i.e.,

for any x , $x' R_X x \geq 0$.

$$\begin{aligned} (x' R_X x &= x' E[x x'] x = E[(x x')(x' x)] \\ &= E[\|x\|^2] \geq 0) \end{aligned}$$

② Their determinants are nonnegative.

③ Their eigenvalues are nonnegative.

Singular Value Decomposition (Eigenvalue Decomposition)

Consider an $n \times n$ square matrix A .

Find its eigenvectors $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n$, and
eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, i.e., $A \underline{e}_i = \lambda_i \underline{e}_i$

$i=1, 2, \dots, n$. (λ_i is the eigenvalue of A corresponding
to eigenvector \underline{e}_i).

$$A \begin{bmatrix} | & | & | & | & | \\ \underline{e}_1 & \underline{e}_2 & \dots & \underline{e}_n \\ | & | & | & | & | \end{bmatrix}_{n \times n} = \begin{bmatrix} | & | & | & | & | \\ \lambda_1 \underline{e}_1 & \lambda_2 \underline{e}_2 & \dots & \lambda_n \underline{e}_n \\ | & | & | & | & | \end{bmatrix}_{n \times n} = \begin{bmatrix} | & | & | & | & | \\ \lambda_1 & 0 & \dots & 0 & \\ | & | & | & | & | \\ 0 & \dots & 0 & \dots & 0 \\ | & | & | & | & | \end{bmatrix}_{n \times n}$$

$\Rightarrow A E = E D$

diag($\lambda_1, \lambda_2, \dots, \lambda_n$)
 $= D$

That is, A can be decomposed into

$$A = E D E'$$

Since $E' E = I = \delta_{ij}$ ($\delta_{ij} = \delta_{ji} = 1$ if $i=j$,
 $\delta_{ij} = 0$ if $i \neq j$) contains orthonormal column vectors.

Since A is symmetric, $\lambda_i \geq 0$, thus,

$$A = E D^{1/2} D^{1/2} E' = \underbrace{(E D^{1/2})}_{B} \underbrace{(E D^{1/2})}'_{B'}$$

where $D^{1/2} = \text{diag}[\lambda_1^{1/2}, \lambda_2^{1/2}, \dots, \lambda_n^{1/2}]$. If

we let $B = E D^{1/2}$, then $A = B B'$.

$$\underline{X} = \underbrace{U D^{1/2}}_{\text{matrix}} \underline{Z} + \underline{M}_X$$

where $G_X = U D U'$ and \underline{Z} is standard

normal. Thus,

$$\underline{X} = \underbrace{\begin{bmatrix} u_1' & u_2' & \dots & u_n' \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{bmatrix}}_{\text{matrix}} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} M_{X1} \\ M_{X2} \\ \vdots \\ M_{Xn} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{d_1} u_1' & \sqrt{d_2} u_2' & \dots & \sqrt{d_n} u_n' \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} M_{X1} \\ M_{X2} \\ \vdots \\ M_{Xn} \end{bmatrix}$$

$$\underline{X} = \sum_{i=1}^n \sqrt{d_i} u_i' z_i + \underline{M}_X$$